LENZ-ISING-ONSAGER PROBLEM IN AN EXTERNAL FIELD AS A SOLUBLE PROBLEM OF MANY FERMIONS

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Abstract

In this paper a new approach to solving the $2D$ and $3D$ Ising models in external magnetic field $H \neq 0$ is developed. The general formalism for the approach to the problem is presented on the example of the $2D$ Ising model in the external magnetic field. The paper presents a new method obtaining the Onsager solution and computations of asymptotic forms of low-temperature free energy for the $2D$ Ising model in the external magnetic field (H). The free energy in the limiting case of the magnetic field tending to zero ($H \to 0, \quad N, M \to \infty$) at arbitrary temperature is also considered ($T \neq 0$).

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I. INTRODUCTION

We will briefly describe the well known model of a magnetic containing(?) variety of spins situated on the vertices of a crystalline lattice. The spin at k can be "up" ($\sigma_k = 1$) or "down" ($\sigma_k = -1$). A microscopic state of the system is characterized by orientations of all the spins. Energy $E\{\sigma\}$ of the microscopic state $\{\sigma\}$ is composed of two contributions, one from the exchange interactions of the spins and described by the interaction constant J_{kl}, and the second from the interaction of the spins with the external magnetic field (H):

$$E\{\sigma\} = -\sum_{kl} J_{kl}\sigma_k\sigma_l - H\sum_{k}\sigma_k, \tag{1.1}$$

where summation is taken over all sites of the lattice. The key problem is calculation of the statistical sum:

$$Z = \sum_{\{\sigma_k\}} \exp(-\beta E\{\sigma\}), \qquad \beta = \frac{1}{k_B T}, \tag{1.2}$$

where T denotes temperature and k_B - the Boltzman constant.

The model described above was introduced by W.Lenz in 1920 [1], and for the one-dimensional case was investigated by E.Ising in 1925 [2]. The first exact solution of the statistical mechanical problem for the $2D$, ($H = 0$) case was found by L.Onsager in 1944 [3]. We use the standard name, the Ising model.

The solution given by Onsager strongly influenced the development of all of statistical physics, and in particular of the theory of phase transitions. It was shown for the first time that exact calculation of the free energy leads to an evidence that thermodynamic quantities behave in the vicinity of the phase transition in a way which is essentially different from that in the approximate models, like e.g. the mean-field theory. The result for spontaneous magnetization $\mathfrak{M}_0$ in the model was presented by Onsager at the conference in Florence in 1949 [4], i.e. 5 years after the successful derivation of the expression for free energy. The first published derivation for $\mathfrak{M}_0$ was given by Yang [5], are recently, alternative derivations have been published both for the free energy and $\mathfrak{M}_0$ [6–9].

In spite of its simplicity, the Ising model is not only non-trivial in higher dimensions ($d \geq 2$), but also it has rich structure. By this we mean not only its connection with other models (for example with the lattice gas models, binary alloys, some models in quantum field theory [7, 10] etc.), and wide application in numerous domains of statistical physics, but also its role as a generator of new ideas and tools, which find its use in various areas of physics and mathematics. There are sufficiently many examples of such applications and we will not discuss them here (some examples can be found in the monograph [11], where stochastic Ising models are considered, and also their connection with Markov processes with local interactions). We would like to stress that this rich structure of the Ising model has maintained a high level of interest in this problem among physicists and mathematicians.

In this paper we present a new approach to the Ising problem in external magnetic field (H), with the nearest-neighbour interaction on the square lattice. In connection with that we would like to mention the paper by Schultz, Mattis and Lieb [6], who applied it to solve the $2D$ Ising model without an external magnetic field. To calculate $\mathfrak{M}_0$ they used a method based on a transfer-matrix using a transformation to a fermionic representation.

This deep, clear and logically closed paper influenced strongly the author and moved him to search for the solution of the problem in external magnetic field. The fundamental idea of the approach of the authors of the paper [6] is transition to a fermionic representation (the transfer-matrix method was essentially used already in the paper by Onsager [3]), and this can be treated in a sense as a problem of interacting fermions on the one-dimensional lattice. In this paper we use essentially the same idea. The difference is the fermionic representation is introduced not on a $1D$ lattice (where the T-matrix is expressed in terms of the Fermi creation and annihilation operators $(c_n^\dagger, c_n)$, [6]) but on a two-dimensional lattice with the doubly indexed Fermi creation - annihilation operators $(c_{nm}^\dagger, c_{nm})$, [12].

II. FORMULATION OF THE PROBLEM

Let us consider the square lattice composed of M columns and N rows, on the vertices of which the quantities σ_{nm} taking one of the two values ± 1 are defined. We will call the quantities the Ising "spins". The multiindex nm numbers the sites of the lattice, where n numbers a row, and m numbers a column. The Ising model with the nearest-neighbour interaction in external magnetic field is given by the Hamiltonian of the form:

$$\mathcal{H} = -J_2 \sum_{nm} \sigma_{nm}\sigma_{n+1,m} - J_1 \sum_{nm} \sigma_{nm}\sigma_{n,m+1} - H \sum_{nm} \sigma_{nm}, \tag{2.1}$$

which takes into account anisotropy in the interaction ($J_{1,2} > 0$) between nearest neighbours, and also the interaction of the spins σ_{nm} with external magnetic field H, directed "up" ($\sigma_{nm} = +1$). The essential problem is calculation of the statistical sum for the system:

$$Z(h) = \sum_{\sigma_{11}=\pm 1} \dots \sum_{\sigma_{NM}=\pm 1} \exp\left(-\beta\mathcal{H}\right)$$

$$= \sum_{(\sigma_{nm}=\pm 1)} \exp\left[\sum_{n,m=1}^{NM} (K_2\sigma_{nm}\sigma_{n+1,m} + K_1\sigma_{nm}\sigma_{n,m+1} + h\sigma_{nm})\right], \tag{2.2}$$

where

$$K_{1,2} = \beta J_{1,2}, \qquad h = \beta H, \qquad \beta = 1/k_B T. \tag{2.3}$$

Periodic boundary conditions are introduced for the variables σ_{nm}. Let us mention here that the statistical sum (2.2) is symmetric with respect to the change $h \to -h$ where h is defined above (2.3).

As is known [6], the statistical sum for the $2D$ Ising model in external field (H) in the representation of second quantization can be written in the form:

$$Z = Tr(V)^N = Tr(V_1 V_2 V_h)^N, \tag{2.4}$$

where the operators V_i, expressed in terms of the Fermi creation and annihilation operators $(c_m^\dagger, c_m)$, are of the form:

$$V_1 = (2\text{sinh}2K_1)^{M/2} \exp\left[-2K_1^* \sum_{m=1}^{M} (c_m^\dagger c_m - 1/2)\right], \tag{2.5}$$

$$V_2 = \exp\left\{K_2\left[\sum_{m=1}^{M-1}(c_m^\dagger - c_m)(c_{m+1}^\dagger + c_{m+1}) - (-1)^{\hat{M}}(c_M^\dagger - c_M)(c_1^\dagger + c_1)\right]\right\}, \tag{2.6}$$

$$V_h = \exp\left\{h\sum_{m=1}^{M}\exp\left[i\pi\sum_{p=1}^{m-1}c_p^\dagger c_p\right](c_m^\dagger + c_m)\right\}, \tag{2.7}$$

where $K_j, (j = 1, 2,)$ and h are defined above (2.3) and $\hat{M} = \sum_1^M c_m^\dagger c_m$ is the operator of the total number of particles and K_1^* and K_1 are connected by the following formulas:

$$\tanh(K_1) = \exp(-2K_1^*), \quad \mathit{or} \quad \sinh 2K_1 \sinh 2K_1^* = 1. \tag{2.8}$$

One can see that the operator V_h in the second quantization representation, that describes interaction of the spins with external magnetic field, has rather complicated structure. It is easy to see that this operator does not commute with the operator $\hat{P} \equiv (-1)^{\hat{M}}$. As a result the operator V_2 has also not a very tractable form, i.e. it has not the needed translational symmetry (2.6). More exactly, although the operators V_1 and V_2 commute with the operator $\hat{P}$, the operator V (2.4) does not commute with the operator $\hat{P}$, i.e. $[\hat{P}, V]_- \neq 0$, because $[\hat{P}, V_h]_- \neq 0$. Therefore, we can not divide all states of the operator $V = V_1V_2V_h$ into eigenstates of the operator $\hat{P}$ with eigenvalues $\lambda = \pm 1$, and this leads to nonconservation of the states with even and odd numbers of fermions (for details see [6]). Namely this is the fundamental reason which stops solving the problem under consideration within this formalism. Nevertheless, the author does not share Ziman's pessimism [13] which is based on some misunderstanding, because he considers actually the approach of the authors of the paper [6], but in the end he writes about limitations of the method of Onsager [3]. In fact Onsager in his approach does not apply the field theoretic language of the creation and annihilation operators as it is in the approach of the authors of the paper [6]. the method of Onsager [3, 14] really shows some limitations when one tries to apply it to solving the $2D$ Ising model in external magnetic field, or for solving the $3D$ Ising model. On the other hand, completely different state of affairs we have for the approach of the authors of the paper [6], where in all its beauty the field theoretic language of the method of second quantization is used. The approach of the authors of the paper [6] allows for generalizations. We intend to present one of such generalizations in this and the following papers devoted to the Ising problem.

Coming back to the difficulties mentioned above which are connected with the operator V_h, (2.7), it is now clear that to overcome the troubles within the approach [6], one should find an appropriate method of substituting the operator V_h (2.7) with another one which would be equivalent to the former in the sense of correct counting of the interaction of external magnetic field with the spins of the system. Namely, as it could be easily seen, the only contribution to Z (2.4) from the operator V_h comes, in the representation of second quantization, from the "even" part with respect to operators $c_m^\dagger, c_m$ of the operator V_h. In principle such transformation could be always done. However, in practice this task seems to be hopeless, and the direct method of calculation of the commutators used by Onsager for solution of the problem without external field here is simply inapplicable. We believe there is not an effective method to do that at least if one stays in the space of given dimension ($d = 2$ for the initial variables σ_{nm}, and $d = 1$ for the variables in the representation of second

quantization $c_m^\dagger$ and c_m). Nevertheless, as we will show below, there is an effective method of transforming the magnetic operator V_h, (2.7), after which the transformed operator allows for the Fourier transform of the operator V (2.4). The idea consists of formulating the problem in the space of higher dimension than the former one, then to pass to the representation of second quantization with the operator V, and afterwards to perform a limit transition with respect to one of the interaction constants, by going with it to zero. Having this done a possibility appears for effective rebuilding of the operator which is responsible for interaction of the spins of the system with external magnetic field. Below we will shortly present this approach on an example of the one-dimensional Ising model which is then applied to solving our basic problem.

III. ONE-DIMENSIONAL ISING MODEL

In the beginning of the consideration of the $1D$ Ising model we have already the complete set of formulas (2.4−8). To apply them to the $1D$ Ising model one should take simply $K_1 = 0$ and $N = 1$. Then, after not complicated transformations, taking into account the expressions (2.8), one can write the following formula for the statistical sum (2.4) $[Z(K_1 = 0) = Z^*]$:

$$Z^* = Tr(V_1^* V_2 V_h), \tag{3.1}$$

where the operators V_2 and V_h are defined above $(2.6 - 7)$, and the operator V_1^* is of the form:

$$V_1^* = \prod_{m=1}^{M} \left[1 + (-1)^{c_m^\dagger c_m}\right]. \tag{3.2}$$

Introducing in an appropriate manner the basis in the representation of occupation numbers [(finite dimensional Fock space): $|0>$ is the vacuum state, $c_m|0>= 0$; $c_m^\dagger|0>$ is a one-particle state $(m = 1, 2, 3, ...)$ etc.], and calculating the trace in (3.1) we get

$$Z^* = \sum_{all(l)} < l|(V_1^* V_2 V_h)|l >= 2^M < 0|(V_2 V_h)|0 >, \tag{3.3}$$

where on the left hand side of (3.3) the summation is over all states $|l>$. It is easy to see that all the matrix elements $< l|(...)|l >$ in (3.3) are equal to zero thanks to the phase factors $(-1)^{c_m^\dagger c_m}$ entering the operator V_1^*, with the exception of the vacuum matrix element $< 0|(...)|0 >$. For this matrix element contribution from the operator (3.2) is equal simply 2^M. From this we obtain the right hand side of the equality (3.3).

Let us mention now that the operators V_h, (2.7), can be represented in the form:

$$V_h = \cosh^M(h) \prod_{m=1}^{M} \left[1 + \psi_m (c_m^\dagger + c_m) \tanh h\right], \tag{3.4}$$

where the phase factor ψ_m is defined in an obvious way (2.7) and we applied the following identity

$$\exp(\rho t) = \cosh t + \rho \sinh t, \qquad \rho^2 = 1.$$

Now, "dragging" the operator V_h , (3.4) through the ket-vector $|0>$, after a number of transformations, we obtain the following representation for $V_h \mid 0>$:

$$V_h \mid 0 >= \cosh^M(h) \prod_{m=1}^{M} e^{\alpha c_m^\dagger} \mid 0 >, \quad \alpha \equiv \tanh(h). \tag{3.5}$$

Deriving the formula (3.5) we dragged all phase factors ψ_m entering the operator V_h through the vacuum state $|0>$ and omitted the annihilation operators c_m, because $c_m \mid 0 >= 0$. We will omit below, for brevity, the ket-vector $|0>$. This should not lead to misunderstandings. Further we not that the operators c_m^+ and c_k^+ commute with the commutator $[c_m^\dagger, c_k^\dagger] = 2c_m^\dagger c_k^\dagger$. As a result, using the Hausdorff- Baker formula ($\alpha, \beta = const$):

$$\exp(\alpha x)\exp(\beta y) = \exp(\alpha x + \beta y + (\alpha\beta/2)[x, y]), \tag{3.6}$$

after $(M-1)$ times application of this formula (3.6) to the operator (3.5), this operator can be represented in the form:

$$V_h = \cosh^M(h)\exp\left[\alpha \sum_{m=1}^{M} c_m^\dagger\right]\exp\left[\alpha^2 \sum_{m=1}^{M}\sum_{p=1}^{M-m} c_m^\dagger c_{m+p}^\dagger\right], \tag{3.7}$$

where α is defined above (3.5). Since all terms in the operator V_2, (2.6) contain bilinear products of the Fermi operators, and the following equality is satisfied

$$\exp\left(\alpha \sum_{m=1}^{M} c_m^\dagger\right) = 1 + \alpha \sum_{m=1}^{M} c_m^\dagger,$$

it is easy to see that in the pairings $< 0 \mid (...) \mid 0 >$ the components linear in $c_m^\dagger$ give null contribution. As a result, we can write the following expression ($V_h \rightarrow V_h^*$) for the operator V_h, (3.7):

$$V_h^* = \cosh^M(h)\exp\left[\alpha^2 \sum_{m=1}^{M}\sum_{p=1}^{M-m} c_m^\dagger c_{m+p}^\dagger\right]. \tag{3.8}$$

Now one can easily see tat the operator $\hat{P} = (-1)^{\hat{M}}$, ($\hat{M} = \sum_1^M c_m^\dagger c_m$) commutes with the operators V_2 and V_h^*, and, as a consequence, the states with even or odd numbers of fermions are conserved. Hence, the statistical sum Z^*, (.3), can be represented in the form:

$$Z^* = 2^M < 0 \mid (V_2^\pm V_h^*) \mid 0 >, \tag{3.9}$$

where

$$V_2^\pm = \exp\left[K_2 \sum_{m=1}^{M} (c_m^\dagger - c_m)(c_{m+1}^\dagger + c_{m+1})\right], \tag{3.10}$$

and the $(+)$ sign in V_2^+ corresponds to the even states, to which are assigned the antiperiodic boundary conditions and the $(-)$ sign to the odd states, to which are assigned the periodic boundary conditions.

Passing in a standard way to the momentum representation

$$c_m = \frac{\exp(-i\pi/4)}{\sqrt{M}} \sum_q e^{iqm}\eta_q,$$

we obtain after some simple transformations on Z^*, (3.9) the expression:

$$Z^* = [2\cosh(h)]^M < 0 \mid (\prod_{0\le q\le\pi} V_{2q}V^*_{hq}) \mid 0 >, \tag{3.11}$$

where

$$V_{2q} = \exp\left\{2K_2\left[(\eta^\dagger_q\eta_q + \eta^\dagger_{-q}\eta_{-q})\cos q + (\eta_q\eta_{-q} + \eta^\dagger_{-q}\eta^\dagger_q)\sin q\right]\right\},$$
$$V^*_{hq} = \exp\left[\alpha^2\left(\frac{1+\cos q}{\sin q}\eta^\dagger_{-q}\eta^\dagger_q + f(q) + f(-q)\right)\right],$$

in which the terms $f(\pm q)$ in the expression for V^*_{hq}:

$$f(q) \equiv -\frac{1+e^{-iq}}{2\sin q}\eta^\dagger_0\eta^\dagger_q$$

and in the case of antiperiodic boundary conditions should be omitted.

Finally, calculating the vacuum matrix element for fixed q, after some not complicated transformations, we get for Z^* (3.11) in the case of even states the expression (Z^*_+):

$$\begin{aligned} Z^*_+ &= [2\cosh(h)]^M \prod_{0<q<\pi} [\cosh 2K_2 - \sinh 2K_2\cos q + \alpha^2\sinh 2K_2(1+\cos q)] \\ &= [2\cosh(h)\cosh K_2]^M \prod_{m=1}^{M}\left[1 + z_2^2 + 2z_2 y - 2z_2(1-y)\cos[\frac{\pi(2m-1)}{M}]\right]^{1/2}, \end{aligned} \tag{3.12}$$

where $z_2 \equiv \tanh K_2$ and $y \equiv \alpha^2 = \tanh^2 h$. Obviously, for N noninteracting Ising models in external magnetic field the statistical sum $W(h)$ is equal to the $N-th$ power of the expression (3.12), i.e. $W(h) = [Z^*_+]^N$. In the case of odd states, as one can easily show the following equality is satisfied:

$$Z^*_- = 2Z^*_+. \tag{3.13}$$

Let us note here that the representation (3.12) unexpectedly finds an application in graph theory. Namely, with help of the representation (3.12) one can calculate the generating function for Hamilton cycles on the simple rectangular lattice ($N\times M$), [15].

Finally, we obtain the following expression for free energy per spin in the thermodynamic limit:

$$-\beta F = \lim_{M\to\infty}\frac{1}{M}\ln Z^* = \ln\left[e^{K_2}\cosh h + (e^{2K_2}\sinh^2 h + e^{-2K_2})^{1/2}\right], \tag{3.14}$$

i.e. the known classic expression [7, 8]. We paid so much attention to the $1D$ Ising model because we wanted to show on the first place to show effectiveness of the proposed method of transformation of the magnetic operator V_h (2.7) to its equivalent (3.8). Additionally, as we mentioned above, a bit different representation of the statistical sum for the $1D$ Ising model (3.11) finds its application in graph theory [15]. Finally, this will help us to save time and place considerably when we will discuss the $2D$ and $3D$ Ising models in external magnetic field.

IV. TRANSFER-MATRIX

In this section we will consider shortly the representation of the statistical sum for the $3D$ Ising model in external magnetic field H, applying for this purpose the well known transfer-martix method [7, 8, 16, 17]. An exhaustive and outstanding presentation of the method the reader can find in the monographs [7, 8], where are also shown other necessary pieces of knowledge on application of this method.

Let us consider a simple cubic lattice consisting of N rows, M columns, and K planes, in the sites of which the "spins" σ_{nmk} are situated, which take on two values: $\sigma_{nmk} = \pm 1$. The Hamiltonian for the $3D$ Ising model in external magnetic field H with the nearest-neighbour interaction is given in the form:

$$\mathcal{H} = - \sum_{(n,m,k)=1}^{NMK} \left(J_1 \sigma_{nmk} \sigma_{n+1,mk} + J_2 \sigma_{nmk} \sigma_{n,m+1,k} + J_3 \sigma_{nmk} \sigma_{nm,k+1} + H \sigma_{nmk} \right), \tag{4.1}$$

where the multiindex (nmk) numbers the sites of the simple cubic lattice $(N \times M \times K)$, and H is the external magnetic field directed "upwards" $(\sigma_{nmk} = +1)$. The constants $(J_j > 0)$ take into account anisotropy of the interaction of the Ising spins. There are periodic boundary conditions imposed, as usual, on the variables σ_{nmk}. The statistical sum for the system $Z_3(h)$ we write in the form:

$$Z_3(h) = \sum_{\sigma_{111}=\pm 1} \dots \sum_{\sigma_{NMK}=\pm 1} e^{-\beta \mathcal{H}} =$$

$$\sum_{\{\sigma_{nmk}=\pm 1\}} \exp \left[\sum_{nmk} \left(K_1 \sigma_{nmk} \sigma_{n+1,mk} + K_2 \sigma_{nmk} \sigma_{n,m+1,k} + K_3 \sigma_{nmk} \sigma_{nm,k+1} + h \sigma_{nmk} \right) \right], \tag{4.2}$$

where the quantities K_i and h are defined as above (2.3), [here and everywhere below summation over nmk (or nm) and also multiplication over nm will mean summation or multiplication over the full set of integer numbers from 1 to N, M and K over each corresponding index, respectively].

In analogy to the two-dimensional case, it is convenient to introduce the notion of k-layer which is understood as a set of Ising spins in all sites of a k-layer:

$$a_{\{nm\}} \equiv a_k = \{\sigma_{nmk}\}, \quad k - \text{fixed}.$$

Then summation in (2.4) can be conveniently executed over the layers a_k, after writing the expression for $Z_3(h)$ in the form:

$$Z_3(h) = \sum_{a_1} \cdots \sum_{a_K} \exp \left\{ \sum_{k=1}^{K} \left[\sum_{nm} (K_1 \sigma_{n+1,mk} + K_2 \sigma_{n,m+1,k} + K_3 \sigma_{nm,k+1} + h) \sigma_{nmk} \right] \right\}$$

$$= \sum_{a_1} \cdots \sum_{a_K} T_{\{\sigma_{nm2}\}}^{\{\sigma_{nm1}\}} T_{\{\sigma_{nm3}\}}^{\{\sigma_{nm2}\}} \cdots T_{\{\sigma_{nm,K+1}\}}^{\{\sigma_{nmK}\}}, \tag{4.3}$$

where

$$T^{\{\sigma_{nmk}\}}_{\{\sigma_{nm,k+1}\}} = \exp\left[\sum_{nm}(K_1\sigma_{n+1,mk} + K_2\sigma_{n,m+1,k} + h)\sigma_{nmk}\right]\exp\left[K_3\sum_{nm}\sigma_{nmk}\sigma_{nm,k+1}\right]. \quad (4.4)$$

We will impose now periodic boundary conditions on the indices n, m, and k, taking

$$\sigma_{N+1,mk} = \sigma_{1mk}, \quad \sigma_{n,M+1,k} = \sigma_{n1k}, \quad \sigma_{nm,K+1} = \sigma_{nm1} \quad (4.5)$$

As a consequence of what was stated above and of the conditions (4.5) we can write $Z_3(h)$ in the form

$$Z_3(h) = Tr(T)^K, \quad (4.6)$$

where T is the transfer-matrix, matrix elements of which are described by equalities (4.4). Matrix elements of the transfer-matrix of the layer-layer Ising model can be written in a bit different form [7], but all these representations are in fact equivalent. Accordingly to the formula (4.4) the matrix T can be represented in the form of a product of the matrices $T_{1,2,3}$ and T_h, each of the same dimension $(2^{NM} \times 2^{NM})$:

$$T = T_3T_2T_1T_h, \quad (4.7)$$

where

$$T_{3,b_{11}\dots b_{NM}}^{\;a_{11}\dots a_{NM}} = \prod_{nm} e^{K_3 a_{nm} b_{nm}}, \quad (4.8)$$

$$T_{2,b_{11}\dots b_{NM}}^{\;a_{11}\dots a_{NM}} = \delta_{a_{11}b_{11}}\dots\delta_{a_{NM}b_{NM}}\prod_{nm} e^{K_2 a_{nm} a_{n,m+1}}, \quad (4.9)$$

$$T_{1,b_{11}\dots b_{NM}}^{\;a_{11}\dots a_{NM}} = \delta_{a_{11}b_{11}}\dots\delta_{a_{NM}b_{NM}}\prod_{nm} e^{K_1 a_{nm} a_{n+1,m}}, \quad (4.10)$$

$$T_{h,b_{11}\dots b_{NM}}^{\;a_{11}\dots a_{NM}} = \delta_{a_{11}b_{11}}\dots\delta_{a_{NM}b_{NM}}\prod_{nm} e^{h a_{nm}}. \quad (4.11)$$

Here we introduced a new way of indexing the matrix elements in the expression (4.4):

$$\{\sigma_{11k}\cdots\sigma_{NMk}\} \equiv \{a_{11}\cdots a_{NM}\}, \;\; \{\sigma_{11,k+1}\cdots\sigma_{NM,k+1}\} \equiv \{b_{11}\cdots b_{NM}\},$$

and we will continue with these assignments till the end of the paper.

Further as is known [16], if we introduce three sets of 2^{NM} - dimensional matrices $(\tau^{x,y,z}_{nm})$ of the form

$$\tau^{x,y,z}_{nm} = 1\otimes 1\otimes\cdots\otimes\tau^{x,y,z}\otimes\cdots\otimes 1\otimes 1, \quad (NM-\text{faktors}), \quad (4.12)$$

where the Pauli matrices $\tau^{x,y,z}$ are situated in these products at the nm-th place, the matrices $T_{1,2,3}$ and T_h, $(4.8-11)$ can be rewritten in the form:

$$T_1 = \exp\left(K_1\sum_{nm}\tau^z_{nm}\tau^z_{n+1,m}\right), \quad T_2 = \exp\left(K_2\sum_{nm}\tau^z_{nm}\tau^z_{n,m+1}\right), \quad (4.13)$$

$$T_3 = (2\sinh 2K_3)^{NM/2} \exp\left(K_3^* \sum_{nm} \tau_{nm}^x\right), \tag{4.14}$$

$$T_h = \exp\left(h \sum_{nm} \tau_{nm}^z\right), \tag{4.15}$$

where the quantities K_3 and K_3^* are connected by the conditions of the form (2.8), and the spin Pauli matrices $\tau_{nm}^{x,y,z}$, (4.12) commute one with each other for different $(nm) \neq (n'm')$, and simultaneously for each given nm these matrices satisfy the standard conditions [16]. It is easy to see that the matrices $T_{1,2,h}$, (4.13, 15) commute one with each other, but they do not commute with the matrix T_3, (4.14). Obviously, for $(h = 0)$ we obtain the known formulas [7] for the matrices $T_{1,2,3}$, describing the three-dimensional Ising model on a simple cubic lattice. To the transition to the $2D$ Ising model in the interaction constants K_1 and K_2 corresponds taking $(K_1 = 0)$ or $(K_2 = 0)$ and simultaneously removal of summation over n $(N = 1)$ or over m $(M = 1)$ respectively. We obtain this way the standard expressions [6, 8] for the $2D$ Ising model in external magnetic field, and the operator T_1 (4.13) is identically equal to the unit operator $(T_1 \equiv 1)$ in the first case, and $(T_2 \equiv 1)$ in the second case, respectively.

A bit different situation occurs in the case of transition to the $2D$ Ising model in the interaction constant K_3. In this case one should take $(K_3 = 0, K = 1)$, i.e. omit summation over k. As a result one can arrive at the following formula for the operator T_3 (4.14):

$$T_3^* \equiv T_3(K_3 = 0) = \prod_{nm}(1 + \tau_{nm}^x). \tag{4.16}$$

Namely this structure of the operator T_3^* enables, finally, effective rebuilding of the magnetic operator T_h (4.15), as it was shown above on the example of the $1D$ Ising model. In this case we can write the expression for the statistical sum for the $2D$ Ising model in the form:

$$Z_2(h) = Tr(T_3^* T_2 T_1 T_h), \tag{4.17}$$

where the matrices $T_{1,2,h}$ are defined by the formulas (4.13, 15), and the matrix T_3^* is defined by the formula (4.16). The advantage of the representation of the statistical sum (4.17) is, in the opinion of the author, in a sense obviously. We will write about this issue additionally below. As it will be clear from what is stated further the matrix $T_2 T_1 T_h$ can be conveniently written in the form $T_h^{1/2} T_2 T_1 T_h^{1/2}$, where we applied commutativity of the factors, following from commutativity of the matrices τ_{nm}^z. The statistical sum (4.17) we rewrite in the form:

$$Z_2(h) = Tr(T_3^* T_h^{1/2} T_2 T_1 T_h^{1/2}), \tag{4.18}$$

where the matrix $T_h^{1/2}$ is defined by the formula

$$T_{h/2} \equiv T_h^{1/2} = \exp\left[(h/2) \sum_{nm} \tau_{nm}^z\right]. \tag{4.19}$$

Below we will use both the expression (4.17) and the representation (4.18), having in mind further applications in graph theory [15, 18].

V. TRANSFORMATION OF T-OPERATOR

A. Introduction of Fermion operators

Schultz, Mattis and Lieb [6] showed that the T-matrix in its standard representation can be expressed in terms of the second quantization Fermi operators. For this aim they applied the known Jordan-Wigner transformations [19] which enable expression of the Fermi operators $(c_m^\dagger, c_m)$ for the one-dimensional system by the Pauli operators $(\tau_m^\pm)$, [8]:

$$c_m = \exp\left(i\pi \sum_{j=1}^{m-1} \tau_j^+ \tau_j^-\right)\tau_m^-, \quad c_m^\dagger = \exp\left(i\pi \sum_{j=1}^{m-1} \tau_j^+ \tau_j^-\right)\tau_m^+. \tag{5.1}$$

As was shown in [12], there is an analogue to the Jordan-Wigner transformations (5.1) which generalizes the former to the two-, three-, and d-dimensional systems.

For this aim we introduce first the following variables [8] to the formulas (4.13 – 16):

$$\tau_{nm}^\pm = \frac{1}{2}(\tau_{nm}^z \pm i\tau_{nm}^y), \tag{5.2}$$

which satisfy anticommutation relations for the same site:

$$\{\tau_{nm}^+, \tau_{nm}^-\}_+ = 1, \qquad (\tau_{nm}^+)^2 = (\tau_{nm}^-)^2, \tag{5.3}$$

and commutation relations for various sites,

$$[\tau_{nm}^\pm, \tau_{n'm'}^\pm]_- = 0, \qquad (nm) \neq (n'm'). \tag{5.4}$$

Quantities $\tau_{nm}^\pm$ are often called Pauli operators. The correspondences

$$\tau_{nm}^x = -2(\tau_{nm}^+\tau_{nm}^- - 1/2), \qquad \tau_{nm}^z = \tau_{nm}^+ + \tau_{nm}^-, \tag{5.5}$$

enable to rewrite the expressions for $T_{1,2,h}$ and T_3^*, (4.13 – 16) in the form:

$$T_1 = \exp\left[K_1 \sum_{nm}(\tau_{nm}^+ + \tau_{nm}^-)(\tau_{n+1,m}^+ + \tau_{n+1,m}^-)\right], \tag{5.6}$$

$$T_2 = \exp\left[K_2 \sum_{nm}(\tau_{nm}^+ + \tau_{nm}^-)(\tau_{n,m+1}^+ + \tau_{n,m+1}^-)\right], \tag{5.7}$$

$$T_h = \exp\left[h \sum_{nm}(\tau_{nm}^+ + \tau_{nm}^-)\right], \tag{5.8}$$

$$T_3^* = \prod_{nm}\left[1 + (1 - 2\tau_{nm}^+\tau_{nm}^-)\right]. \tag{5.9}$$

As it was mentioned above, the Pauli operators $\tau_{nm}^\pm$ behave as Fermi operators when considered for one site, and as Bose operators when considered for different sites. In order

to transform to the fermionic representation, i.e. to the Fermi operators in the whole lattice, we will introduce an analogue of the Jordan-Wigner transformations (5.1), which will enable to express Fermi operators $(c_{nm}^{\dagger}, c_{nm})$ by Pauli operators $\tau_{nm}^{\pm}$ for the two-dimensional system. Namely, there exist in the two-dimensional case two sets of such transformations [12], which we represent here in the form:

$$\alpha_{nm}^{\dagger} = \exp\left(i\pi \sum_{k=1}^{n-1} \sum_{l=1}^{M} \tau_{kl}^{+}\tau_{kl}^{-} + i\pi \sum_{l=1}^{m-1} \tau_{nl}^{+}\tau_{nl}^{-}\right) \tau_{nm}^{+},$$

$$\alpha_{nm} = \exp\left(i\pi \sum_{k=1}^{n-1} \sum_{l=1}^{M} \tau_{kl}^{+}\tau_{kl}^{-} + i\pi \sum_{l=1}^{m-1} \tau_{nl}^{+}\tau_{nl}^{-}\right) \tau_{nm}^{-}, \tag{5.10}$$

and

$$\beta_{nm}^{\dagger} = \exp\left(i\pi \sum_{k=1}^{N} \sum_{l=1}^{m-1} \tau_{kl}^{+}\tau_{kl}^{-} + i\pi \sum_{k=1}^{n-1} \tau_{km}^{+}\tau_{km}^{-}\right) \tau_{nm}^{+},$$

$$\beta_{nm} = \exp\left(i\pi \sum_{k=1}^{N} \sum_{l=1}^{m-1} \tau_{kl}^{+}\tau_{kl}^{-} + i\pi \sum_{k=1}^{n-1} \tau_{km}^{+}\tau_{km}^{-}\right) \tau_{nm}^{-}. \tag{5.11}$$

It is easy to show, using formulas $(5.3-4)$ that the operators $(\alpha_{nm}^{\dagger}, \alpha_{nm})$ and $(\beta_{nm}^{\dagger}, \beta_{nm})$ are Fermi operators in the whole lattice, i.e. they satisfy anticommutation relations for all sites:

$$\{\alpha_{nm}^{\dagger}, \alpha_{nm}\}_{+} = 1, \quad (\alpha_{nm}^{\dagger})^2 = (\alpha_{nm})^2 = 0; \quad \{\alpha_{nm}^{\dagger}, \alpha_{n'm'}^{\dagger}\}_{+} = ... = 0, \quad (nm) \neq (n'm'), \tag{5.12}$$

and analogously for the β-operators. There are also inverse transformations:

$$\tau_{nm}^{+} = \exp\left[i\pi \sum_{k=1}^{n-1} \sum_{p=1}^{M} \alpha_{kp}^{\dagger}\alpha_{kp} + i\pi \sum_{p=1}^{m-1} \alpha_{np}^{\dagger}\alpha_{np}\right] \alpha_{nm}^{\dagger}, \tag{5.13}$$

etc., which can be easily proved by application of the identities

$$\exp\left(i\pi \sum_{nm} \tau_{nm}^{+}\tau_{nm}\right) = \prod_{nm}\left(1 - 2\tau_{nm}^{+}\tau_{nm}^{-}\right) = \prod_{nm} \tau_{nm}^{x},$$

from which one can easily derive the equalities

$$\tau_{nm}^{+}\tau_{nm}^{-} = \alpha_{nm}^{\dagger}\alpha_{nm} = \beta_{nm}^{\dagger}\beta_{nm}. \tag{5.14}$$

The formulas (5.14) express conditions of local equality of the occupation numbers for α- and β- fermions in one site. Further, as it follows from $(5.10-11)$ and (5.13), α- and β-operators are connected by canonical non-linear transformations:

$$\alpha_{nm}^{\dagger} = \exp(i\pi\varphi_{nm})\beta_{nm}^{\dagger}, \quad \alpha_{nm} = \exp(i\pi\varphi_{nm})\beta_{nm},$$

$$\varphi_{nm} = \left[\sum_{k=n+1}^{N} \sum_{p=1}^{m-1} + \sum_{k=1}^{n-1} \sum_{p=m+1}^{M} \right] \alpha^{\dagger}_{kp}\alpha_{kp} = [\cdots]\beta^{\dagger}_{kp}\beta_{kp}, \tag{5.15}$$

where the operators φ_{nm} obviously commute with the operators $(\alpha^{\dagger}_{nm}, \alpha_{nm})$ and $(\beta^{\dagger}_{nm}, \beta_{nm})$, i.e.

$$[\varphi_{nm}, \alpha^{\dagger}_{nm}]_{-} = \cdots = \cdots = [\varphi_{nm}, \beta_{nm}]_{-} = 0. \tag{5.16}$$

Commutation relations among α- and β- operators are more complicated. Namely, as one can check by direct calculation that the following commutation relations hold:

$$\{\alpha^{\dagger}_{nm}, \beta_{nm}\}_{+} = \{\beta^{\dagger}_{nm}, \alpha_{nm}\}_{+} = (-1)^{\varphi_{nm}}, \tag{5.17}$$

$$[\alpha_{nm}, \beta_{n'm'}]_{-} = \ldots = [\alpha^{\dagger}_{nm}, \beta^{\dagger}_{n'm'}]_{-} = 0, \qquad \begin{pmatrix} n' \le n-1, \ m' \ge m+1 \\ n' \ge n+1, \ m' \le m-1 \end{pmatrix}, \tag{5.18}$$

and

$$\{\alpha_{nm}, \beta_{n'm'}\}_{+} = \ldots = \{\alpha^{\dagger}_{nm}, \beta^{\dagger}_{n'm'}\}_{+} = 0, \tag{5.19}$$

in all other cases, where φ_{nm} are defined above (5.15). This way we get rather specific structure of commutation relations among α- and β- operators in the lattice, although this structure shows some symmetry. Here is the right place to compare the situation described above with the situation we get using the second quantization method. For a system composed of different particle one introduces the second quantization operators of different kinds for different particles. The operators connected to either bosons or fermions satisfy the standard commutation relations. As far as the operators for different fermions are concerned, it is usually assumed without any proof [20], that within the limits of nonrelativistic theory they could be treated formally as commuting or anticommuting. Both assumptions lead to the same results when the second quantization method is applied. Nevertheless, in the relativistic theory, which allows for transmutations of various particles we should consider creation and annihilation operators for different fermions as anticommuting. On the other hand, in our case we deal formally with "quasiparticles" of the α- and β- types underlying separately the Fermi statistics with commutation relations among particles of different types being however dependent on relative position of these "quasiparticles" in the sites of lattice. Such a situation, as far as is known to the author, was not present in earlier works on application of the second quantization method.

B. The $T_{1,2,h}$- and T^{*}_{3} - operators

Before writing the T- operators $(5.6-9)$ in terms of Fermi operators, let as make a few remarks. First, the operator T^{*}_{3} (5.9) can be expressed in terms of α as well as β-operators, because of (5.14):

$$T^{*}_{3} = \prod_{nm} \left[1 + (-1)^{\alpha^{\dagger}_{nm}\alpha_{nm}} \right] = \prod_{nm} \left[1 + (-1)^{\beta^{\dagger}_{nm}\beta_{nm}} \right], \tag{5.20}$$

where the basic in the Fock representation should be chosen as to be expressed in terms of the α- or β- operators, respectively. Second, the operators $T_{1,2,h}$ we can also express in terms of either α- or β- operators. Nevertheless, the operator T_2 we write in terms of the α- operators and the operator T_1 we write in terms of the β- operators for reasons which will become clear later.

Now, due to $(.10-11)$ we can write the operator T_h (5.8) in the form:

$$T_h = \exp\left[h\sum_{nm}\theta_{nm}(\alpha^\dagger_{nm}+\alpha_{nm})\right] = \exp\left[h\sum_{nm}\psi_{nm}(\beta^\dagger_{nm}+\beta_{nm})\right], \tag{5.21}$$

where θ_{nm} is defined as the first factor in (5.13), and ψ_{nm} is defined by:

$$\psi_{nm} = \exp\left[i\pi\sum_{k=1}^{N}\sum_{p=1}^{m-1}\beta^\dagger_{kp}\beta_{kp} + i\pi\sum_{k=1}^{n-1}\beta^\dagger_{km}\beta_{km}\right].$$

Transformation of the operators $T_{1,2}$ is a bit more complicated. Taking into account cyclic boundary conditions (4.5), we will write first a sequence of equalities analogous to (5.14):

$$\tau^+_{N,m}\tau^+_{1,m} = -(-1)^{\hat N_m}\beta^\dagger_{N,m}\beta^\dagger_{1,m}, \qquad \tau^+_{N,m}\tau^-_{1,m} = -(-1)^{\hat N_m}\beta^\dagger_{N,m}\beta_{1,m},$$

$$\tau^-_{N,m}\tau^+_{1,m} = (-1)^{\hat N_m}\beta_{N,m}\beta^\dagger_{1,m}, \qquad \tau^-_{N,m}\tau^-_{1,m} = (-1)^{\hat N_m}\beta_{N,m}\beta_{1,m}, \tag{5.22}$$

and

$$\tau^+_{n,M}\tau^+_{n,1} = -(-1)^{\hat M_n}\alpha^\dagger_{n,M}\alpha^\dagger_{n,1}, \qquad \tau^+_{n,M}\tau^-_{n,1} = -(-1)^{\hat M_n}\alpha^\dagger_{n,M}\alpha_{n,1},$$

$$\tau^-_{n,M}\tau^+_{n,1} = (-1)^{\hat M_n}\alpha_{n,M}\alpha^\dagger_{n,1}, \qquad \tau^-_{n,M}\tau^-_{n,1} = (-1)^{\hat M_n}\alpha_{n,M}\alpha_{n,1}, \tag{5.23}$$

where

$$\hat g_n \equiv (-1)^{\hat M_n}, \quad \hat M_n = \sum_{m=1}^{M}\alpha^\dagger_{nm}\alpha_{nm}; \qquad \hat g_m \equiv (-1)^{\hat N_m}, \quad \hat N_m = \sum_{n=1}^{N}\beta^\dagger_{nm}\beta_{nm}, \tag{5.24}$$

which can be obtained by using the formulas $(5.10-13)$. Therefore we can write the following representations for the operators $T_{1,2}$:

$$T_1 = \exp\left\{K_1\sum_{m=1}^{M}\left[\sum_{n=1}^{N-1}(\beta^\dagger_{nm}-\beta_{nm})(\beta^\dagger_{n+1,m}+\beta_{n+1,m}) - \hat g_m(\beta^\dagger_{Nm}-\beta_{Nm})(\beta^\dagger_{1,m}+\beta_{1,m})\right]\right\}, \tag{5.25}$$

$$T_2 = \exp\left\{K_2\sum_{n=1}^{N}\left[\sum_{m=1}^{M-1}(\alpha^\dagger_{nm}-\alpha_{nm})(\alpha^\dagger_{n,m+1}+\alpha_{n,m+1}) - \hat g_n(\alpha^\dagger_{nM}-\alpha_{nM})(\alpha^\dagger_{n,1}+\alpha_{n,1})\right]\right\}. \tag{5.26}$$

Finally, let us express the operator T_2 in terms of the β -operators:

$$T_2=\exp\left\{K_2\sum_{n=1}^{N}\left[\sum_{m=1}^{M-1}\hat{\chi}_{nm}(\beta^\dagger_{nm}-\beta_{nm})(\beta^\dagger_{n,m+1}+\beta_{n,m+1})-\hat{G}\hat{\chi}_{nM}(\beta^\dagger_{nM}-\beta_{nM})(\beta^\dagger_{n,1}+\beta_{n,1})\right]\right\}, \tag{5.27}$$

where the operators $\hat{G}$ and $\hat{\chi}_{nm}$ are defined by the formulas:

$$\hat{G}\equiv\exp\left[i\pi\sum_{nm}\alpha^\dagger_{nm}\alpha_{nm}\right]=\exp\left[i\pi\sum_{nm}\beta^\dagger_{nm}\beta_{nm}\right]=(-1)^{\hat{S}},$$

$$\hat{\chi}_{nm}=\exp\left[i\pi\sum_{k=n+1}^{N}\beta^\dagger_{km}\beta_{km}+i\pi\sum_{k=1}^{n-1}\beta^\dagger_{k,m+1}\beta_{k,m+1}\right], \tag{5.28}$$

and we applied the relations analogous to (5.23), but expressed in terms of the β- operators. The operator $\hat{S}$ introduced above (5.28) is the operator of the number of particles, which is connected with the operators $\hat{N}$ and $\hat{M}$, (5.24) by relations:

$$\hat{S}=\sum_{n=1}^{N}\hat{M}_n=\sum_{m=1}^{M}\hat{N}_m,\qquad \hat{G}=\prod_{n=1}^{N}\hat{g}_n=\prod_{m=1}^{M}\hat{g}_m. \tag{5.29}$$

It is easy to see that the operator $\hat{G}$, (5.28) commutes with the operators T_1 and T_2 (5.25−27), but it does not commute with the operator T_h (5.21), because the following relations are satisfied:

$$\{\hat{G},\alpha^\dagger_{nm}\}_+=\cdots=\{\hat{G},\beta_{nm}\}_+=0 \tag{5.30}$$

Of course, we can also express the operators T_1 and T_h in terms of the α- operators and we can write down the formulas if they are necessary.

It was shown above (4.17) that the statistical sum for the $2D$ Ising model in external magnetic field can be represented by the trace of the operator T, which was expressed here by the Fermi second quantization operators. Introducing, as in the one-dimensional case, a basic in the occupation numbers representation [20] for the α- and β- fermions (2^{NM} - dimensional space in the Fock representation), and calculating then appropriate matrix elements $< l|T|l >$, it is easy to see that because of multiplicative character of the operator T_3^*, (5.20) all matrix elements, besides the vacuum matrix element $< 0|T|0 >$, are equal to zero. For the vacuum matrix element contribution from the operator T_3^* is equal simply to 2^{NM}, and we can write $Z_2(h)$ $(4.17-18)$ in the following form:

$$Z_2(h)=2^{NM}<0|(T_2T_1T_h)|0>=2^{NM}<0|(T_{h/2}T_2T_1T_{h/2})|0>, \tag{5.31}$$

where the vacuum state $|0>$ is defined in the standard manner

$$\alpha_{nm}|0>=\beta_{nm}|0>=0,\qquad n(m)=1,2,...,N(M), \tag{5.32}$$

and operators $T_{1,2,h}$ are defined by the formulas (5.21) and $(5.25-27)$. Let us stress that the vacuum state (5.32) for the α- and β- fermions can differ among themselves at most by

a constant phase factor, which in the given case can always be taken to be equal to unity. However, it is no longer true in the case of multiparticle states for the α- and β- fermions, because in this case the essential role begin to play phase factors $(-1)^{\varphi_{nm}}$, (5.15). As an exception serve the one-particle states, for which, as it can be easily found from (5.15), we have:

$$\alpha^{\dagger}_{nm}|0>= (-1)^{\varphi_{nm}}\beta^{\dagger}_{nm}|0>= \beta^{\dagger}_{nm}|0>,$$

for all(nm). In all other cases the α- and β- states will differ one from each other by their sign which depends on indices (nm) of the corresponding sites. This very fact implies main difficulty in the proposed approach to solving the problem under consideration. This difficulty can be, however, avoided.

Let us make two remarks here. It is obvious that the representation (55.31) for the statistical sum $Z_2(h)$ does not depend on the kind of variables (α- or β- operators) with which we introduce the basic in the representation of occupation numbers, because equality of local occupation numbers (5.14) holds for the α- and β- fermions. Further, we expressed the operator T_2 in terms of the α- and β- variables $(5.26-27)$, although we will work mainly with the expression (5.26). The reason is that in the representation (5.27) for T_2 the operators $\hat{\chi}_{nm}$ are present (5.28). They are phase factors and it is difficult in practice to remove them. Difficulty coming from the presence of these operators is of the same kind, which was found by the authors of the paper [6] who considered the case with external magnetic field. Simultaneously, the representation (5.26) for T_2 does not involve the phase factors, justifying the choice. Nevertheless, the expression (5.27) for T_2 will be necessary in the analysis of the boundary conditions, which play here important role. Analogous statement applies to the operator T_1, which we expressed in terms of the α- and β- variables (5.25) and which also does not contain phase factors of the type of $\hat{\chi}_{nm}$. The essence of our approach lies in the structure of the transformations $(5.10-11)$, which allows for expression of the operators $T_{1,2}$ in the form (5.26), which does not contain the phase factors.

Now, we transform the magnetic operator T_h (5.21), or more exactly the ket-vector $T_h|0>$, entering the expression (5.31) for $Z_2(h)$. Exactly in the sense one should understand the equivalence of the two operators T_h and T_h^*, acting on the vacuum state $|0>$. Below we will omit $|0>$, as this should not lead to misunderstandings. Analogously, we introduce the notation $T_{h/2}^{l,r}$ for the transformed bra-vector $<0|T_{h/2}$ and the transformed ket-vector $T_{h/2}|0>$, respectively, omitting further bra- and ket-vectors of the vacuum state $(<0|, |0>)$. Continuing with considerations analogous to these, which gave us the expression (3.7) in the one-dimensional case, the operator T_h (5.21) we represent in the form:

$$T_h=(\cosh h)^{NM}\exp\left[\alpha\sum_{nm}\beta^{\dagger}_{nm}\right]\exp\left\{\alpha^2\left[\sum_{n,n'}\sum_{m=1}^{M}\sum_{p=1}^{M-m}\beta^{\dagger}_{nm}\beta^{\dagger}_{n',m+p}+\sum_{n=1}^{N}\sum_{k=1}^{N-n}\sum_{m=1}^{M}\beta^{\dagger}_{nm}\beta^{\dagger}_{n+k,m}\right]\right\}, \tag{5.33}$$

where $\alpha\equiv\tanh h$. Analogously, the operators $T_{h/2}^{l,r}$ we write in the form:

$$T^l_{h/2}=(\cosh\frac{h}{2})^{NM}\exp\left[\mu\sum_{nm}\alpha_{nm}\right]\exp\left\{\mu^2\left[\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{p=1}^{M-m}\alpha_{n,m+p}\alpha_{nm}+\sum_{n=1}^{N}\sum_{k=1}^{N-n}\sum_{m,m'}^{M}\alpha_{n+k,m'}\alpha_{nm}\right]\right\}, \tag{5.34}$$

$$T^r_{h/2}=(\cosh\frac{h}{2})^{NM}\exp\left[\mu\sum_{nm}\beta^\dagger_{nm}\right]\exp\left\{\mu^2\left[\sum_{n,n'}^{N}\sum_{m=1}^{M}\sum_{p=1}^{M-m}\beta^\dagger_{nm}\beta^\dagger_{n',m+p}+\sum_{n=1}^{N}\sum_{k=1}^{N-n}\sum_{m=1}^{M}\beta^\dagger_{nm}\beta^\dagger_{n+k,m}\right]\right\}, \tag{5.35}$$

where $\mu\equiv\tanh(h/2)$. The operators T_h and $T^{l,r}_{h/2}$ are of rather complicated structure. However, they do not contain the phase factors any longer. Substituting the expressions $(5.33-35)$ to the equalities (5.31), the statistical sum $Z_2(h)$ can be written in the form:

$$Z_2(h)=2^{NM}<0|(T_2T_1T^*_h)|0>, \tag{5.36}$$

or

$$Z_2(h)=2^{NM}<0|(T^*_lT_2T_1T^*_r+\mu^2AT^*_lT_2T_1T^*_rB)|0>\equiv 2^{NM}<0|(U_1+U_2)|0>, \tag{5.37}$$

where the operators $U_{1,2}$ are defined in the obvious way, and the operators T^*_h and $T^*_{l,r}$ are given by the formulas $(5.33-35)$, in which one should omit the factors

$$\exp\left(\alpha\sum_{nm}\beta^\dagger_{nm}\right),\quad \exp\left(\mu\sum_{nm}\alpha_{nm}\right),\quad \exp\left(\mu\sum_{nm}\beta^\dagger_{nm}\right),$$

and the operators A and B are of the form:

$$A=\sum_{nm}\alpha_{nm},\qquad B=\sum_{nm}\beta^\dagger_{nm}. \tag{5.38}$$

In derivation of $(5.36-37)$ we used the fact that the diagonal matrix elements for the product of odd number of Fermi operators is equal to zero, and that the following equalities are true

$$\exp\left[a\sum_{nm}\alpha_{nm}(\beta^\dagger_{nm})\right]=1+a\sum_{nm}\alpha_{nm}(\beta^\dagger_{nm}),$$

where a is a c-valued function. One should remember also that the operators $T_{1,2}$ $(5.25-27)$ contain only bilinear products of the Fermi operators. With this ends the rebuilding procedure for the magnetic operator.

C. The Boundary Conditions

With the aim of further simplification of the operators $T_{1,2}$ we should consider boundary conditions for the α- and β- operators, taking periodic boundary conditions for the Pauli operators $\tau^\pm_{nm}$ (5.2) in both indices (nm) as a starting point. Let us shortly discuss this problem here. First, since all terms in $T_{1,2}$ and T^*_h contain bilinear products of the Fermi operators, the following formulas are valid:

$$[\hat{G},\ T_1]_-=[\hat{G},\ T_2]_-=[\hat{G},\ T^*_h]_-=\ 0, \tag{5.39}$$

which shows that the states with even or odd number of fermions are preserved as well for the α- as for the β- particles. The operator $\hat{G}$, entering (5.39), is defined above (5.28). Analogously, the following formulas are true:

$$[\hat{G},\ U_1]_- \ =\ [\hat{G},\ U_2]_- =\ 0,$$

where the operators $U_{1,2}$ are defined above. However, the operators $\hat{g}_n$ and $\hat{g}_m$ (5.24) do not commute with the operators $T = T_2 T_1 T_h^*$ or $U = U_1 + U_2$ (5.37), and this fact implies some difficulties. First, let us note that the following equalities hold:

$$\hat{G} = \prod_n \hat{g}_n = \prod_m \hat{g}_m, \qquad \lambda_{\hat{G}} = \prod_n \lambda_{\hat{g}_n} = \prod_m \lambda_{\hat{g}_m}, \tag{5.40}$$

where by $\lambda_{\hat{G}}$, $\lambda_{\hat{g}_n}$ and $\lambda_{\hat{g}_m}$ we denoted eigenvalues of the operators $\hat{G}$, $\hat{g}_n$ and $\hat{g}_m$, equal ± 1.

Let us consider first the case corresponding to the state with even number of fermions ($\lambda_{\hat{G}} = +1$). In this case, as one can easily see, we should choose antiperiodic boundary conditions for the β- operators with respect to the second index m (for all n), and antiperiodic boundary conditions for the α- operators with respect to the first index n (for all m), i.e.

$$\beta^\dagger_{n,M+1} = -\beta^\dagger_{n,1}, \quad \beta_{n,M+1} = -\beta_{n,1}, \quad (n = 1, 2, ..., N);$$

$$\alpha^\dagger_{N+1,m} = -\alpha^\dagger_{1,m}, \quad \alpha_{N+1,m} = -\alpha_{1,m}, \quad (m = 1, 2, ..., M). \tag{5.41}$$

Then the boundary conditions for the β- operators with respect to the first index n depend on $\hat{g}_m$, and the boundary conditions for the α- operators with respect to the second index m depend on $\hat{g}_n$. More exactly, it depends on at which step we fix the eigenvalues $\lambda_{\hat{g}_m}$ and $\lambda_{\hat{g}_n}$, respectively. The only limitations on the choice of the eigenvalues and corresponding boundary conditions give equalities (5.40). The whole freedom in the choice of boundary conditions consist of 2^N possible boundary conditions for the α- operators in their second index, and 2^M possible boundary conditions for the β- operators in their first index. Detailed analysis shows that we can without loosing generality choose homogeneous boundary conditions, i.e. antiperiodic boundary conditions for the α- operators in their second index m, which corresponds to ($\lambda_{\hat{g}_n} = +1$) for each n, and antiperiodic boundary conditions for the β- operators in their first index n, which corresponds to ($\lambda_{\hat{g}_m} = +1$) for each m, i.e.

$$\alpha^\dagger_{n,M+1} = -\alpha^\dagger_{n,1}, \quad \alpha_{n,M+1} = -\alpha_{n,1}, \quad \lambda_{\hat{g}_n} = +1, \ \ (n = 1, 2, ...N);$$

$$\beta^\dagger_{N+1,m} = -\beta^\dagger_{1,m}, \quad \beta_{N+1,m} = -\beta_{1,m}, \quad \lambda_{\hat{g}_m} = +1, \ \ (m = 1, 2, ..., M);$$

$$\lambda_{\hat{G}} = \prod_n \lambda_{\hat{g}_n} = (+1)^N = \prod_m \lambda_{\hat{g}_m} = (+1)^M = +1, \tag{5.42}$$

for each parity of the numbers N and M. Analogously, one can show that in the case of the odd states ($\lambda_{\hat{G}} = -1$) the boundary conditions for the α- and β- operators can be written in the form:

$$\alpha^\dagger_{N+1,m} = +\alpha^\dagger_{1,m}, \quad \alpha_{N+1,m} = +\alpha_{1,m}, \quad (m = 1, 2, ..., M);$$

$$\alpha^\dagger_{n,M+1} = +\alpha^\dagger_{n,1}, \quad \alpha_{n,M+1} = +\alpha_{n,1}, \quad \lambda_{\hat{g}_n} = -1, \ \ (n = 1, 2, ...N);$$

$$\beta^\dagger_{N+1,m} = +\beta^\dagger_{1,m}, \quad \beta_{N+1,m} = +\beta_{1,m}, \quad \lambda_{\hat{g}_m} = -1, \ \ (m = 1, 2, ..., M);$$

$$\beta^\dagger_{n,M+1} = +\beta^\dagger_{n,1}, \quad \beta_{n,M+1} = +\beta_{n,1}, \quad (n = 1, 2, ..., N);$$

$$\lambda_{\hat{G}} = \prod_n \lambda_{\hat{g}_n} = (-1)^N = \prod_m \lambda_{\hat{g}_m} = (-1)^M = -1, \tag{5.43}$$

for N and M odd. It is obvious that the constraints on parity of N and M are not important here, because we can always choose N and M in the form $(N = 2N' + 1, \ \ M = 2M' + 1)$, and then go to infinity with N' and M' independently.

One can show exactly that the boundary conditions for the α- and β- operators chosen this way are not contradictory, if we take into account simultaneously the conditions of local equality of the occupation numbers for the α- and β- fermions (5.14). As a result we can write down the following expressions for the operators $T_{1,2}$ $(5.25 - 26)$:

$$T_1^\pm = \exp\left[K_1 \sum_{n,m=1}^{N,M} (\beta^\dagger_{nm} - \beta_{nm})(\beta^\dagger_{n+1,m} + \beta_{n+1,m})\right], \tag{5.44}$$

$$T_2^\pm = \exp\left[K_2 \sum_{n,m=1}^{N,M} (\alpha^\dagger_{nm} - \alpha_{nm})(\alpha^\dagger_{n,m+1} + \alpha_{n,m+1})\right], \tag{5.45}$$

where the upper sign $(+)$ corresponds to the states with even numbers of fermions $(\lambda_{\hat{G}} = +1)$, and the lower sign $(-)$ corresponds to the states with odd numbers of fermions $(\lambda_{\hat{G}} = -1)$, with the appropriate boundary conditions $(5.41 - 43)$. This way, the form of the operators $T_{1,2}$ for the even and odd states is preserved. What is changing is only the boundary conditions.

VI. THE PARTITION FUNCTION

In this section we perform all the calculations for the statistical sum written in the form (5.36), and in the end of the section we only give the results for $Z_2(h)$ written in the form (5.37) symmetric in the magnetic operator. This can be done almost automatically, since all calculations in the latter case are analogous to the ones given below.

Collecting all the results derived above, we can write the following expression for the statistical sum (5.36):

$$Z_2(h) = (2\cosh h)^{NM} < 0|T|0 >, \quad T \equiv T_2^\pm T_1^\pm T_h^*\ , \tag{6.1}$$

where the operators T_h^* and $T_{1,2}^\pm$ are defined by the formulas (5.33) and $(5.44 - 45)$. In the formula (5.33) one should only omit the factor $\exp(\alpha \sum_{nm} \beta^\dagger_{nm})$, and move the constant factor $(\cosh h)^{NM}$ out of the expression $< 0|(...)|0 >$. Let us remind before the diagonalization of the T- operator in (6.1), in which the multiplicative components $T_{1,2}^\pm$ and T_h^* are expressed by the Fermi operators of the α- and β- types, that these operators satisfy the mixed commutative relations $(5.17 - 19)$. As a result also their Fourier transforms will satisfy in general rather complex commutative relations.

A. Momentum Representation

Let us pass now to the momentum representation:

$$\alpha^\dagger_{nm} = \frac{\exp(i\pi/4)}{(NM)^{1/2}} \sum_{q,p} e^{-i(nq+mp)} \xi^\dagger_{qp}, \qquad \beta^\dagger_{nm} = \frac{\exp(i\pi/4)}{(NM)^{1/2}} \sum_{q,p} e^{-i(nq+mp)} \eta^\dagger_{qp}, \tag{6.2}$$

where the factor $\exp(i\pi/4)$ was introduced for convenience. Antiperiodic boundary conditions give: $e^{iqN} = -1, \;\; e^{ipM} = -1$, where

$$q(p) = \;\; \pm\frac{\pi}{N(M)}, \quad \pm\frac{3\pi}{N(M)}, \quad \pm\cdots, \tag{6.3}$$

and the periodic boundary conditions give: $e^{iqN} = 1, \;\; e^{ipM} = 1$, where

$$q(p) = 0, \quad \pm\frac{2\pi}{N(M)}, \quad \pm\frac{4\pi}{N(M)}, \quad \pm\cdots. \tag{6.4}$$

Substituting (6.2) into (5.33) and $(5.44-45)$ we get after straight forward transformations the following expressions for the operators $T^\pm_{1,2}$ i T^*_h:

$$\begin{aligned} T^\pm_1 = \exp\Bigg\{ 2K_1 \sum_{0\le q,p\le\pi} \Big[&(\eta^\dagger_{qp}\eta_{qp} + \eta^\dagger_{q,-p}\eta_{q,-p} + \eta^\dagger_{-qp}\eta_{-qp} + \eta^\dagger_{-q,-p}\eta_{-q,-p}) \cos q + \\ &(\eta^\dagger_{-q,-p}\eta^\dagger_{q,p} + \eta^\dagger_{-q,p}\eta^\dagger_{q,-p} + \eta_{q,p}\eta_{-q,-p} + \eta_{q,-p}\eta_{-q,p}) \sin q \Big] \Bigg\} = \prod_{0\le q,p\le\pi} T^\pm_1(q,p), \end{aligned} \tag{6.5}$$

$$\begin{aligned} T^\pm_2 = \exp\Bigg\{ 2K_2 \sum_{0\le q,p\le\pi} \Big[&(\xi^\dagger_{qp}\xi_{qp} + \xi^\dagger_{q,-p}\xi_{q,-p} + \xi^\dagger_{-qp}\xi_{-qp} + \xi^\dagger_{-q,-p}\xi_{-q,-p}) \cos p + \\ &(\xi^\dagger_{-q,-p}\xi^\dagger_{q,p} + \xi^\dagger_{q,-p}\xi^\dagger_{-q,p} + \xi_{q,p}\xi_{-q,-p} + \xi_{-q,p}\xi_{q,-p}) \sin p \Big] \Bigg\} = \prod_{0\le q,p\le\pi} T^\pm_2(q,p), \end{aligned} \tag{6.6}$$

$$T^*_h = \exp\left\{ \sum_{0\le q,p\le\pi} \Big[\alpha(h,q)(\eta^\dagger_{-q,-p}\eta^\dagger_{q,p} + \eta^\dagger_{-q,p}\eta^\dagger_{q,-p}) \Big] + \Phi(h) \right\} = \prod_{0\le q,p\le\pi} T^\pm_h(q,p), \tag{6.7}$$

where $\alpha(h,q) \equiv \tanh^2 h \frac{1+\cos q}{\sin q}$.

In the formulas $(6.5-7)$ the upper sign $(+)$ corresponds to the case of even states, for which one should omit the term $\Phi(h)$ in the formula (6.7), and the lower sign $(-)$ corresponds to the case of odd states with respect to the operator of the total number of particles $(\hat{S})$. The function $\Phi(h)$ is of arbitrary complicated form and we will not write it down here. We only mention that it plays an analogous role to the role played by the function $f(q)$ in the one-dimensional case (3.11). We used commutativity of the operators $T^\pm_{1,2}(q,p)$ and $T^\pm_h(q,p)$ for different (q,p) to write the expressions $(6.5-7)$. Namely,

$$[T^\pm_1(q,p), T^\pm_1(q',p')]_- = [T^\pm_2(q,p), T^\pm_2(q',p')]_- = [T^\pm_h(q,p), T^\pm_h(q',p')]_- = 0,$$

as can be easily verified. The statistical sum (6.1) we rewrite now in the form:

$$Z_2^{\pm}(h) = (2\cosh h)^{NM} < 0| \left[\prod_{0\le q,p\le\pi} T_2^{\pm}(q,p) \right] \left[\prod_{0\le q,p\le\pi} T_1^{\pm}(q,p) T_h^{\pm}(,p) \right] |0>, \tag{6.8}$$

where $|0>$ - the function of the fermionic vacuum in the space of occupation numbers in the momentum representation. This function was denoted in the same way as in the "coordinate" representation but this should not lead to any misunderstandings. We used also commutativity of the operators $T_1^{\pm}(q,p)$ and $T_h^{\pm}(q,p)$ for $(q,p) \neq (q',p')$ to write the formula (6.8). The operators $(\xi_{qp}, \xi_{qp}^{\dagger})$ and $(\eta_{qp}, \eta_{qp}^{dag})$ satisfy the standard commutation relations. On the other hand the commutation relations mixing them are of rather complex form, in contrast to the relations in the "coordinate" representation $(5.17-19)$. This, by the way, is the cause of the lack of commutativity of the operators $T_2^{\pm}(q,p)$ and $T_1^{\pm}(q,p)T_h^{\pm}(q,p)$ for $(q,p) \neq (q',p')$. Now we will maximally simplify the bra-vector $< 0|(...)$ and the ket-vector $(...)|0>$, which are present in the expression (6.8) for $T_2^{\pm}(h)$, "transferring" the corresponding operators through the vacuum state.

Now, we will consider in some details the case corresponding to the even number of fermions ($\lambda_{\hat{G}} = +1$) which means the choice of the antiperiodic boundary conditions (6.3). In the end of the paper we will shortly consider the case of the odd states ($\lambda_{\hat{G}} = -1$) to which correspond the periodic boundary conditions (6.4). First, let us note that the following equality holds

$$\sum_{q,p} \xi_{qp}^{\dagger}\xi_{qp} \;=\; \sum_{q,p} \eta_{qp}^{\dagger}\eta_{qp} \,. \tag{6.9}$$

Further, it is obvious that for fixed (q,p) the quantities $T_2^{\pm}(q,p)$ and $T_1^{\pm}(q,p)T_h^{\pm}(q,p)$ are represented by the matrices of the size (16×16), each of which is considered in its own space of states $\mathcal{P}_{\xi}$ and $\mathcal{P}_{\eta}$, respectively. After introduction of the bases, each of which is built of 16 functions:

$$\Phi_0 \equiv |0>_{\xi}, \qquad \Phi_{q,p} = \xi_{qp}^{\dagger}\Phi_0, \qquad \Phi_{-q,-p;\,q,p} = \xi_{-q,-p}^{\dagger}\xi_{qp}^{\dagger}\Phi_0, \;\; \cdots \tag{6.10}$$

$$\Psi_0 \equiv |0>_{\eta}, \qquad \Psi_{q,p} = \eta_{qp}^{\dagger}\Psi_0, \qquad \Psi_{-q,-p;\,q,p} = \eta_{-q,-p}^{\dagger}\eta_{qp}^{\dagger}\Psi_0, \;\; \cdots, \tag{6.11}$$

where Φ_0 and Ψ_0 are the functions of Fermi vacuum (which, as was mentioned above, we denoted by $\Phi_0 = \Psi_0 = |0>$), we obtain after a sequence of transformations the expression for the statistical sum $Z_2^{+}(h)$ in the case of the even states:

$$Z_2^{+}(h) = (2\cosh h)^{NM} \left(\prod_{0<q,p<\pi} A_1^2(q,h) \right) \left(\prod_{0<q,p<\pi} A_2^2(p) \right) < 0|\tilde{T}_2^{+}\tilde{T}_1^{+}(h)|0>, \tag{6.12}$$

where

$$\tilde{T}_1^{+}(h) = \exp\left[\sum_{0<q,p<\pi} B_1(q)(\eta_{-q,-p}^{\dagger}\eta_{q,p}^{\dagger} + \eta_{-q,p}^{\dagger}\eta_{q,-p}^{\dagger}) \right], \tag{6.13}$$

$$\tilde{T}_2^{+} = \exp\left[\sum_{0<q,p<\pi} B_2(p)(\xi_{q,p}\xi_{-q,-p} + \xi_{-q,p}\xi_{q,-p}) \right], \tag{6.14}$$

and the quantities $A_1(q,h)$, ... , $B_2(p)$ are defined by the formulas:

$$A_1(q,h) = \cosh 2K_1 - \sinh 2K_1 \cos q + \alpha(h)\sinh 2K_1 \sin q,$$
$$A_2(p) = \cosh 2K_2 - \sinh 2K_2 \cos p, \quad \alpha(h) = \tanh^2 h \frac{1+\cos q}{\sin q},$$
$$B_1(q) = \frac{\alpha(h)[\cosh 2K_1 + \sinh 2K_1 \cos q] + \sinh 2K_1 \sin q}{A_1(q,h)}, \quad B_2(p) = \frac{\sinh 2K_2 \sin p}{A_2(p)}. \quad (6.15)$$

Analogously, we can after a sequence of transformations, similar to given above, get the following expression of the statistical sum understood in the form (5.37) symmetrical with respect to the parameter of the external magnetic field h:

$$Z_2^+(h) = (2\cosh^2 \frac{h}{2})^{NM} \left(\prod_{0<q,p<\pi} C_1^2(q)\right)\left(\prod_{0<q,p<\pi} C_2^2(p)\right) < 0|V_2^+(h/2)V_1^+(h/2)|0 >, \quad (6.16)$$

where

$$V_1^+(h/2) = \exp\left[\sum_{0<q,p<\pi} D_1(q,h/2)(\eta^\dagger_{-q,-p}\eta^\dagger_{q,p} + \eta^\dagger_{-q,p}\eta^\dagger_{q,-p})\right], \quad (6.17)$$
$$V_2^+(h/2) = \exp\left[\sum_{0<q,p<\pi} D_2(p,h/2)(\xi_{q,p}\xi_{-q,-p} + \xi_{-q,p}\xi_{q,-p})\right], \quad (6.18)$$

and the quantities $C_1(q,h/2)$, ... , $D_2(p,h/2)$ are defined by the formulas:

$$C_1(q,h/2) = \cosh 2K_1 - \sinh 2K_1 \cos q + \alpha(h,q)\sinh 2K_1 \sin q,$$
$$C_2(p,h/2) = \cosh 2K_2 - \sinh 2K_2 \cos p + \alpha(h,p)\sinh 2K_2 \sin p,$$
$$D_1(q,h/2) = \frac{\alpha(h,q)[\cosh 2K_1 + \sinh 2K_1 \cos q] + \sinh 2K_1 \sin q}{C_1(q,h/2)},$$
$$D_2(p,h/2) = \frac{\alpha(h,p)[\cosh 2K_2 + \sinh 2K_2 \cos p] + \sinh 2K_2 \sin p}{C_2(p,h/2)}, \quad (6.19)$$

where $\alpha(h,x) = \tanh^2(h/2)(1+\cos x)/(\sin x)$. For the purpose of derivation of the expressions $(6.16-19)$ we applied the fact for the even states the operator U_2 in (5.37) gives vanishing contribution to the statistical sum $Z_2^+(h)$. We gave here two representations (6.12) and (6.16) for $Z_2^+(h)$, because as can be shown they both can be applied in the graph theory [15] as we mentioned above.

In principle, one could now expand the vacuum matrix element in (6.12) or in (6.16) into a sum of vacuum matrix elements of the type

$$< 0|\xi_{q,p}\xi_{-q,-p}\cdots\eta^\dagger_{-q',-p'}\eta^\dagger_{q',p'}\cdots|0 >,$$

derive appropriate commutation relations for the ξ- and $\eta^\dagger$-operators, and, finally, sum up the series. Nevertheless in practise this task seems to be extremely difficult, as believes the author. Therefore we will proceed the other way. Namely, we will come back to the "coordinate" representation, i.e. to the α- and β- operators. Then the operators $\tilde{T}^+_{1,2}$ $(6.13-14)$ or $V^+_{1,2}$ $(6.17-18)$ are expressed as follows:

$$V_1^+ = \exp\left[\sum_{n=1}^{N}\sum_{l=1}^{N-n}\sum_{m=1}^{M} a(l)\beta^\dagger_{nm}\beta^\dagger_{n+l,m}\right],$$
$$V_2^+ = \exp\left[\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{k=1}^{M-m} b(k)\alpha_{n,m+k}\alpha_{nm}\right], \tag{6.20}$$

where $a(l)$ and $b(k)$ are given by:

$$a(l) = \frac{1}{N}\sum_{0<q<\pi} 2D_1(q)\sin(lq), \quad b(k) = \frac{1}{M}\sum_{0<p<\pi} 2D_2(p)\sin(kp), \tag{6.21}$$

for the "symmetric" case, and by:

$$c(l) = \frac{1}{N}\sum_{0<q<\pi} 2B_1(q)\sin(lq), \quad d(k) = \frac{1}{M}\sum_{0<p<\pi} 2B_2(p)\sin(kp), \tag{6.22}$$

for the "nonsymmetric" case, where the quantities $B_1(q)$, ... , $D_2(p)$ are defined above by (6.15) and (6.19). Here we used in both cases the same notation $V^+_{1,2}$, and further we continue with this convention. As can be seen from $(6.13-14)$ and $(6.17-18)$, the structure of the operators $\tilde{T}^+_{1,2}$ in the "coordinate" representation is the same as in the case (6.20). The only change concerns the weight factors: $a(l) \to c(l)$ i $b(k) \to d(k)$. The whole procedure used above corresponds to the renormalization of the interaction constants in the former expression (5.31) for the statistical sum. We will be exploring this topic more thoroughly in the following papers of this series. Moreover, here appears also a delicate problem of the boundary conditions, connected with the expressions (6.20). The discussion of this problem we also postpone to a future publication. Here we mention only that in the thermodynamic limit we can neglect the boundary effects. On the other hand, in the situation at hands it is much easier and more convenient to consider the diagram representation for the vacuum matrix element $<0|V_2^+V_1^+|0>$ in the "coordinate" representation than in the "momentum" one, which we denote here by S, i.e.

$$S = <0|V_2^+V_1^+|0> \equiv <0|G|0>. \tag{6.23}$$

B. The Diagram Representation for S

Our aim now is to calculate the vacuum matrix element S (6.23) for the sum of products of Fermi creation and annihilation operators. The operator G entering (6.23) is a polynomial in the variables $a(l)$, $b(k)$, α_{nm} and $\beta^\dagger_{nm}$. Since G enters in the (6.23) expectation value form $<0|G|0>$, not all terms in the polynomial give a different from zero contribution to the matrix element S. Expanding G and substituting the expansion into (6.23), the quantity S can be represented in the form of the sum of the vacuum matrix elements $\sum_\nu S_\nu$, where S_ν is the vacuum matrix element for the ν-th term of the polynomial G. As it follows from (6.20), all terms of the polynomial G are products of various pairs $b(k)\alpha_{n,m+k}\alpha_{nm}$ and $a(l)\beta^\dagger_{nm}\beta^\dagger_{n+l,m}$, which we will call below α- pairs and β- pairs. Obviously, all the terms in the polynomial G with non-equal numbers of the α- and β- pairs give vanishing contribution. Moreover, not all terms in the polynomial G with equal numbers of the α- and β- pairs

will give nonvanishing contribution to S. Namely, the non-zero contribution to S will give only these terms with equal numbers of the α- and β- pairs, in which each annihilation operator α_{nm} is paired with the corresponding creation operator $\beta^{\dagger}_{n'm'}$ with identical indices $(n = n', m = m')$. In the opposite case this term gives obviously no contribution to S.

This way we arrive at a diagrammatic representation by noticing that to each vacuum matrix element S_ν we can uniquely assign a set of lines (links), connecting some of the sites of the lattice. For example, to the graphs at $Fig.1, a) - d)$ correspond the following matrix elements:

$$
\begin{aligned}
&a) \quad a(2)b(3) < 0|\alpha_{n,m+3}\alpha_{nm}\beta^{\dagger}_{n,m+3}\beta^{\dagger}_{n+2,m+3}|0> ;\\
&b) \quad a^2(1)a(2)b^2(1)b(2) < 0|\alpha_{n,m+1}\alpha_{nm}\alpha_{n+1,m+1}\alpha_{n+1,m-1}\alpha_{n+2,m}\alpha_{n+2,m-1}\\
&\qquad \times\beta^{\dagger}_{n+1,m-1}\beta^{\dagger}_{n+2,m-1}\beta^{\dagger}_{nm}\beta^{\dagger}_{n+2,m}\beta^{\dagger}_{n,m+1}\beta^{\dagger}_{n+1,m+1}|0> ;\\
&c) \quad a^2(1)a^2(4)b^2(1)b^2(2) < 0|\alpha_{n,m+1}\alpha_{nm}\alpha_{n+1,m+1}\alpha_{n+1,m}\alpha_{n+1,m-2}\alpha_{n+1,m-4}\alpha_{n+5,m-2}\alpha_{n+5,m-4}\\
&\qquad \times\beta^{\dagger}_{n+1,m-4}\beta^{\dagger}_{n+5,m-4}\beta^{\dagger}_{n+1,m-2}\beta^{\dagger}_{n+5,m-2}\beta^{\dagger}_{nm}\beta^{\dagger}_{n+1,m}\beta^{\dagger}_{n,m+1}\beta^{\dagger}_{n+1,m+1}|0> ;\\
&d) \quad a^2(2)a(4)b(2)b(3)b(5) < 0|\alpha_{n,m+2}\alpha_{nm}\alpha_{n+2,m}\alpha_{n+2,m-3}\alpha_{n+4,m+2}\alpha_{n+4,m-3}\\
&\qquad \times\beta^{\dagger}_{n+2,m-3}\beta^{\dagger}_{n+4,m-3}\beta^{\dagger}_{nm}\beta^{\dagger}_{n+2,m}\beta^{\dagger}_{n,m+2}\beta^{\dagger}_{n+4,m+2}|0> \ .
\end{aligned}
\tag{6.24}
$$

As one can see from the formulas (6.20) and (6.24), to each horizontal line of the "length" k corresponds the factor $b(k)$. Also, to each vertical line of the "length" l corresponds the factor $a(l)$. The $a(l)$ and $b(k)$ are defined by the expressions (6.22) for the nonsymmetric case. As was shown above a nonzero contribution to S give only these matrix elements S_ν, which do not contain equal numbers of the α- and β- pairs. Moreover, the necessary condition for a non-zero contribution is the annihilation operators α_{nm} pair with the corresponding creation operators $\beta^{\dagger}_{nm}$. Geometrically this condition means that from the whole family of possible graphs only those for which in each site meet under "right angle" only zero or 2 lines (links) give a non-zero contribution to S. In other words, the graphs in any site of which meet two horizontal or two vertical lines are forbidden. The simplest examples of such graphs are shown in $Fig.1, b) - e)$. As a result all graphs giving non-vanishing contribution to$S$ should be closed. Moreover, in each site of the graphs selfintersections of lines (links) are forbidden, since $(\alpha_{nm})^2 = (\beta^{\dagger}_{nm})^2 = 0$. ¿From the point of view of the graph theory to the closed graphs described above correspond nonoriented Hamilton cycles (with valency of sites $\delta = 0, 2$) on the simple rectangular lattice [18, 23, 24].

This way the vacuum matrix element S (6.23) can be represented in the form

$$
S = \sum_{\nu} S_\nu = \sum [sum \ \ on \ \ all \ \ closed \ \ graphs], \tag{6.25}
$$

where in the calculations every multiple-connected graph is counted as one (for example, the graph in the $Fig.1, c)$). Every closed graph gives the contribution equal to

$$
(\pm 1) \prod_{j=1}^{s} a(l_j)b(k_j), \tag{6.26}
$$

where s is the number of the horizontal links, which is equal to the vertical links. Further, applying the connection between the α- and β- operators $(5.15-19)$, and the Wick theorem [21, 22], one can show that any vacuum matrix element giving non-zero contribution into

the sum S (6.25), can be split into a product of the matrix elements, corresponding to the connected parts of the graph (which we will call below for shortness the simple loops without selfintersections in the sites of the lattice). We can check by direct computation, using the commutation relations $(5.17-19)$ for the α- and β- operators, that for example the graphs from the $Fig.1, (b-d)$ contribute with the sign $(+)$. Other graphs can contribute with the sign $(-)$ as well as, for example, the graph in $Fig.1, e$. Commutation relations for the α- and β- operators $(5.17-19)$ are illustrated in an appealing way in the $Fig.2$, where the distinguished operator α_{nm} $(*)$ for the fixed site (nm) commutes with the β- operators in the sites $(n'm')$, signed with the cross $(\times)$. For all others sites the α- and β- operators anticommute. As a result the contribution from each particular graph splits into a product of contributions from the simple loops. The contribution from a simple loop with s horizontal and s vertical links is equal to:

$$\mathcal{L}_s = (\pm 1) \prod_{j=1}^{s} a(l_j) b(k_j) \ .$$

The expression for S (6.25) is now of the form:

$$S = 1 + \sum_{\{s\}} \mathcal{L}_s + \sum_{\{s\},\{q\}} \mathcal{L}_s \mathcal{L}_q + \ldots \equiv \Gamma^{(h)}(z_1, z_2, y) \ ,$$

where $a(l_j)$ and $b(k_j)$ are functions of $z_1 \equiv \tanh K_1$, $z_2 \equiv \tanh K_2$ and $y \equiv \tanh^2(h/2)$ for the symmetric case, and $y \equiv \tanh^2 h$ in the case asymmetric with respect to the parameter of the external magnetic field (h). A contribution to (6.28) gives besides summation over the number of links s, also the summation over all lengths of these links $\{k\}$ and $\{l\}$, for fixed s. As can be easily seen, the summation in (6.28) over the lengths of the horizontal $\{k\}$ and vertical $\{l\}$ links is performed independently. In the graph theory [18, 23] the function (6.28) is called the generating function, as we mentioned above, introducing for it the notation $\Gamma^{(h)}(z_1, z_2, y)$, where the upper index (h) means being a member of the set of Hamilton cycles. The problem was reduced this way to the summation over all Hamilton cycles with the varying length of the step (edge) on the rectangular lattice of the type described above.

Now, let us note that the graph representation of $Z_2(h)$, described above, looks similar to the diagrammatic representation for the statistical sum of the $2D$ Ising model in the vanishing magnetic field $(h = 0)$, (see, e.g., the papers [25–27]). In this case, as is known [25], the statistical sum can be represented in the form:

$$Z(K_1, K_2) = (2 \cosh K_1 \cosh K_2)^{NM} \sum_{\alpha,\beta} g_{\alpha,\beta} \tanh^{\alpha} K_1 \tanh^{\beta} K_2 \ , \tag{6.27}$$

where $g_{\alpha,\beta}$ denotes the number of the closed graphs consisting of β horizontal and α vertical links. Since these links connect the closest sites of the square lattice, to each link α is assigned the factor (weight) $\tanh K_1$, and to each link β is assigned the factor $\tanh K_2$. In some sites of the graph a simple selfintersection is possible, i.e. in one site of the graph meet zero, two, or four lines. This corresponds to the nonoriented Euler cycles of the degree $\delta \leq 4$, [18, 24]. In the $Fig.3$ is shown one of the simplest graphs contributing to the sum (6.29) for $Z_2(K_1, K_2)$. The essential difference of this case in comparison with the case with

the field (h), described by us above, lies in the latter property, because in our case in one site of the lattice can meet only zero or two lines (horizontal and vertical). This corresponds, as was discussed above, to the nonoriented Hamilton cycles on the square lattice [18, 24]. The second difference is that the α- and β- links in (6.28) can connect not only the nearest sites of the lattice. This result in the appearance of dependence of the weight factors $a(l_j)$ and $b(k_j)$ on the distances l and k between the sites of the lattice in the vertical and the horizontal direction, respectively. As we mentioned above, the problem of calculation of the sum (6.28) can be called in the language of the graph theory [18] the problem of summation over the Hamilton cycles (simple cycles) on the rectangular lattice with $(N \times M)$ sites with varying "length" of the edges in the horizontal and in the vertical directions, respectively. Simultaneously, the problem of the sum (6.29) is equivalent to the problem of summation over all possible Euler cycles, described above of the type ($\delta \leq 4$) on the same lattice. As is known [18], there is a close correspondence between the Euler and the Hamilton graphs. For some types of the Euler graphs one can consider instead the corresponding Hamilton graphs. The reversed statement is not true. In the papers [15] is shown one more example of the nontrivial connection between the generating functions for the Euler cycles and the Hamilton cycles on the simple rectangular lattice. Namely, in the papers [15] was shown that the generating function $\Gamma^{(h)}(z_1, z_2, y = 0)$ for the Hamilton cycles described above is exactly equal to the generating function $\Gamma^{(e)}(z_1, z_2)$ for the Euler cycles ($\delta \leq 4$) for the $2D$ Ising model [18]. Therefore, the following equality is true:

$$\Gamma^{(h)}(z_1, z_2) = \prod_{n=1}^{N} \prod_{m=1}^{M} \left[(1+z_1^2)(1+z_2^2) - 2z_1(1-z_2^2)\cos\frac{2\pi n}{N} - 2z_2(1-z_1^2)\cos\frac{2\pi m}{M}\right]^{\frac{1}{2}}, \qquad (6.28)$$

where $z_{1,2} \equiv \tanh K_{1,2}$. Taking in (6.16) the external magnetic field to be equal to zero ($h = 0$), and using the equality (6.30) we arrive at the classical expression [3] for the free energy on one Ising spin in the $2D$ Ising model. Let us note that the contribution of each graph (connected or disconnected), which consists of a set of the Hamilton cycles, can be represented in the form of a product of the determinants of the incidence matrices B_ν:

$$(\pm) \prod_{\nu}^{\omega} \det |B_\nu| \,,$$

where ω denotes the order of connectedness of the graph under consideration. It is equal to the number of the simple loops creating the graph. This way we conclude that for the computation of the statistical sum for the $2D$ Ising model in the external magnetic field it is necessary to calculate the generating functions for the Hamilton graphs on the simple rectangular lattice of the type described above (see [15], [28], [29]).

VII. LIMITING CASES

A. The Onsager solution

Let us shortly discuss one of the method of receiving the Onsager solution [29]. Putting in Eqs.(6.16) and (6.19) the magnetic field equal to zero ($h = 0$), the partition function Z_2 (6.16) takes the form:

$$Z_2 = 2^{NM}[(1-z_1^2)(1-z_2^2)]^{-\frac{NM}{2}} < 0|T_2^* T_1^*|0 >, \tag{7.1}$$

where $z_{1,2} = \tanh{(K_{1,2})}$, while operators $T^*_{1,2}$ can be written in the "coordinate" representation as:

$$T_1^* = \exp\left\{\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{l=1}^{N-n} z_1^l \beta^\dagger_{nm}\beta^\dagger_{n+l,m}\right\}, \qquad T_2^* = \exp\left\{\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{k=1}^{M-m} z_2^k \alpha_{n,m+k}\alpha_{nm}\right\}. \tag{7.2}$$

The Ward–Kac solution [30], briefly described in [31], contains a topological considerations. Namely, for a given closed graph (we consider here Euler graphs on a lattice) a factor $\alpha = \exp(i\pi/4)$ is added to a left turn, and a factor $\alpha^{-1} = \exp(-i\pi/4)$ to a right turn. Closed graphs (i.e. which we want to include) are thus taken into account and forbidden graphs are compensated if me follow various paths over these graphs. Full proof of this theorem was given by Sherman [32]. Similar result holds for hamiltonian graphs on a lattice with variable length described above which will be shown in simple cases below. However, we will follow the methods of [27, 33] in our consideration.

First of all let us mention that some of hamiltonian loops (e.g. $Fig.1, e$) contribute with minus $(-)$ sign in formula (6.28) for S. Namely straightforward verification, with the help of commutation relations $(5.17-18)$ shows that each doubly intersecting link of the one shown in $Fig.1, e$ contributes a minus sign to a overall sign of a simple loop (6.27) for all admissible diagrams. At the same time each "simple double link" of the type shown in $Fig.1, f$ contributes a $(+)$ sign to the overall sign of a simple loop (6.27). All other simple loops without "double links" of the shown in $Fig.1, b--d$ come with a plus $(+)$ sign in the sum (6.28). (Let us note that there is a one to one correspondence between Euler graphs on a lattice and hamiltonian graphs with variable step without "double links", the hamiltonian graph may contain one, two or more simple loops. In order to establish this correspondence it is necessary to select in the Euler graph all intermediate vertices together with intersecting horizontal and vertical links of the Euler graph.)

It is easy to understand now, that if in expression (6.28) for S all simple loops are taken with a $(+)$ sign, all left (and right) turns in a simple loop give a factor $\alpha = \exp(i\pi/4)$, $(\alpha^{-1} = \exp(-i\pi/4))$, than the problem of calculating the sum for S (6.28) is in fact reduced to a "random walk" on a lattice with variable step [27, 31, 33]. In fact, with such a way of following simple loops all loops with "double links" cancel (e.g. loops in $Fig.1, e$ and d), as it should be. In this way one can follow all the loops with "double links" and verify that they cancel each other. Moreover, one can check using various examples, following the same reasoning as given in [27, 31, 33] that if we follow various paths over all hamiltonian loops with variable step without "double link" (including relevant weights α and α^{-1} at each turn) than all the allowed diagrams will cancel. One should stress here that such full cancellation of forbidden diagrams in every order takes place only in the case of factorizable weights (z_1^l, z_2^k) corresponding to step lengths l and k, respectively.

Returning to our problem and using the results of [27, 33], we obtain for S (6.28) the following expression:

$$S = \exp[-\sum_{r=1}^{\infty} f_r], \tag{7.3}$$

where f_r - sum over all single loops with length $(r = 2s)$, i.e. consisting of s horizontal and s vertical links. Each horizontal line contributes a factor $(z_2^k e^{i\varphi/2})$, and each vertical line -

a factor $(z_1^l e^{i\varphi/2})$, where angle $(\varphi = \pm\pi/2)$ corresponds to left or right turn. Introducing quantity $W_r(n,m,\nu)$ - sum over all possible paths with number of links equal to $(r = s_1 + s_2)$ from a given initial point (n_0, m_0, ν_0) to a point (n,m,ν), where ν - auxiliary index, corresponding to four directions $(1,2,3,4)$ on a square lattice, we get for f_r:

$$f_r = \frac{1}{2r} \sum_{n_0, m_0, \nu_0} W_r(n_0, m_0, \nu_0). \tag{7.4}$$

One can easily get the following recursion relations for $W_r(n,m,\nu)$ with $(\alpha \equiv \exp(i\pi/4))$:

$$\begin{aligned}
W_{r+1}(n,m,1) &= 0 + \alpha^{-1} \sum_{l=1}^{N} z_1^l W_r(n-l,m,2) + 0 + \alpha \sum_{l=1}^{N} z_1^l W_r(n+l,m,4); \\
W_{r+1}(n,m,2) &= \alpha \sum_{k=1}^{M} z_2^k W_r(n,m-k,1) + 0 + \alpha^{-1} \sum_{k=1}^{M} z_2^k W_r(n,m+k,3) + 0; \\
W_{r+1}(n,m,3) &= 0 + \alpha \sum_{l=1}^{N} z_1^l W_r(n-l,m,2) + 0 + \alpha^{-1} \sum_{l=1}^{N} z_1^l W_r(n+l,m,4); \\
W_{r+1}(n,m,4) &= \alpha^{-1} \sum_{k=1}^{M} z_2^k W_r(n,m-k,1) + 0 + \alpha \sum_{k=1}^{M} z_2^k W_r(n,m+k,3) + 0.
\end{aligned} \tag{7.5}$$

The meaning of recursion relations (7.5) is evident. Since the point $(n,m,1)$ can be reached from $(n',m,2)$ and $(n'',m,4)$; i.e. from above and from below (direction "1" was chosen to be "right"), where $n' = n - l, n'' = n + l$, and l ranges, strictly speaking, from 1 to $N-1$. However, for large N the summation over l can be extended to N, which was done in expression (7.5), because in the thermodynamic limit these boundary conditions do not play a role. Hamiltonian structure of simple loops is evident in the structure of recursion relations (7.5), which should be compared to the case of Euler graphs [31, 33]. Writing the relations (7.5) in matrix form

$$W_{r+1}(n,m,\nu) = \sum_{n',m',\nu'} \Lambda(n,m,\nu|n',m',\nu') W_r(n',m',\nu'), \tag{7.6}$$

one can easily see that the following relation holds:

$$Tr\Lambda^r = \sum_{n_0,m_0,\nu_0} W_r(n_0,m_0,\nu_0), \tag{7.7}$$

and also

$$f_r = \frac{1}{2r} Tr\Lambda^r = \frac{1}{2r} \sum_i \lambda_i^r. \tag{7.8}$$

Taking into account (7.4) and (3.6) we get for S, (7.3) the following relation:

$$S = \prod_i \sqrt{1 - \lambda_i}, \tag{7.9}$$

where λ_i - eigenvalue of the matrix $\Lambda(n,m,\nu)$, $(i = 1,2,...,4NM)$. The matrix $\Lambda(n,m,\nu)$ can be easily diagonalized over indices (n,m) with the help of Fourier transformation:

$$W_r(n,m,\nu) = \sum_{q,p=0}^{N,M} e^{\frac{2\pi i}{N}nq+\frac{2\pi i}{M}mp} W_r(q,p,\nu). \tag{7.10}$$

Inserting (7.10) into (7.5), for fixed (q,p) we get:

$$\Lambda(q,p,\nu|q,p,\nu') = \begin{bmatrix} 0 & \alpha^{-1}\sum_{l=1}^N z_1^l \varepsilon^{-lq} & 0 & \alpha \sum_{l=1}^N z_1^l \varepsilon^{lq} \\ \alpha \sum_{k=1}^M z_2^k \omega^{-kp} & 0 & \alpha^{-1}\sum_{k=1}^M z_2^k \omega^{kp} & 0 \\ 0 & \alpha \sum_{l=1}^N z_1^l \varepsilon^{-lq} & 0 & \alpha^{-1}\sum_{l=1}^N z_1^l \varepsilon^{lq} \\ \alpha^{-1}\sum_{k=1}^M z_2^k \omega^{-kp} & 0 & \alpha \sum_{k=1}^M z_2^k \omega^{kp} & 0 \end{bmatrix}, \tag{7.11}$$

where $\alpha \equiv \exp(i\pi/4)$, $\varepsilon \equiv \exp(2\pi i/N)$, $\omega \equiv \exp(2\pi i/M)$.

It is evident, that for fixed (q,p) it suffices to calculate the determinant of (4×4) matrix:

$$\prod_{j=1}^{4}(1-\lambda_j) = Det(\delta_{\nu\nu'} - \Lambda_{\nu\nu'}) \equiv A(q,p), \tag{7.12}$$

and after simple calculations for $A(q,p)$, (7.12) we get the following formula:

$$A(q,p) = \frac{(1+z_1^2)(1+z_2^2) - 2z_1(1-z_2^2)\cos(2\pi q/N) - 2z_2(1-z_1^2)\cos(2\pi p/M)}{(1-2z_1\cos(2\pi q/N)+z_1^2)(1-2z_2\cos(2\pi p/M)+z_2^2)}. \tag{7.13}$$

In (7.13) we have neglected the terms proportional to z_1^N and z_2^M, since for large N and M , $z_1^N \approx 0$ i $z_2^M \approx 0$, for $z_{1,2} < 1$. Finally for asymptotically large (N,M) for S (7.9) with the help of (7.13) we get:

$$S = \prod_i \sqrt{1-\lambda_i} = \prod_{q,p=0}^{N,M} A^{1/2}(q,p)$$

$$= \prod_{q,p=0}^{N,M} \left[\frac{(1+z_1^2)(1+z_2^2) - 2z_1(1-z_2^2)\cos(2\pi q/N) - 2z_2(1-z_1^2)\cos(2\pi p/M)}{(1-2z_1\cos(2\pi q/N)+z_1^2)(1-2z_2\cos(2\pi p/M)+z_2^2)}\right]^{1/2}. \tag{7.14}$$

Of course, for asymptotically large (N,M) the expression (7.14) goes to expression (6.30), because of following relations

$$\prod_{q=0}^{N}(1-2z_1\cos\frac{2\pi q}{N}+z_1^2) = 1, \quad \prod_{p=0}^{M}(1-2z_2\cos\frac{2\pi p}{N}+z_2^2) = 1,$$

for $(N,M \to \infty)$, $z_{1,2} < 1$. Finally, using (6.23) and inserting (7.14) into formula (7.1), for free energy per Ising spin in the thermodynamic limit we get the well known Onsager solution [3]. The method of finding the Onsager solution, given in this paper, disregarding its complications, allows for analytical study of the Ising-Onsager problem in external magnetic field in several limiting cases in two and three dimensions. The proposed method of receiving the Onsager solution, as well as previously known graphical methods, work only for case $(a(l) = z_1^l,\ \ b(k) = z_2^k,\ \ l(k) = 1,2,...)$. Can be shown that all these methods are not applicable if faktors $(a(l)$ and $b(k))$ have different functional structure. Anyhow, contrary to all previously derived methods (graphical et al.) the presented method allows, in such or other approximation, for accounting external magnetic field H.

B. Low-temperature asymptotic for $F_{2D}(h)$

The aim of this chapter is to consider the free energy per one Ising spin in the external magnetic field for some limit cases. For that reason the parameters $(K_{1,2}, h)$ are to be renormalised in the following way $(K_{1,2} \geq 0)$:

$$\sinh 2K^*_{1,2} = \beta_{1,2}[\sinh 2K_{1,2}(1 - \tanh^2(h/2)],$$
$$\cosh(2K^*_{1,2}) = \beta_{1,2}[\cosh 2K_{1,2} + \tanh^2(h/2) \sinh 2K_{1,2}],$$
$$\beta_{1,2} = [1 + 2\tanh^2(h/2) \sinh 2K_{1,2} e^{2K_{1,2}}]^{-1/2}, \quad \tanh^2 h^*_{1,2} = \tanh^2(h/2) \frac{\beta_{1,2} \exp(2K_{1,2})}{\cosh^2 K^*_{1,2}} \tag{7.15}$$

The above presented formulae are adequate for symmetrical case. For asymmetric case it is sufficient to substitute, for instance, $K^*_2 \to K_2, h^*_2 \to 0$. In short, in this case only the parameter K_1 and the field h are subjects of renormalisation. Formulae (6.21) and (6.22) included in (6.20), take the form:

$$a(l) = z_1^{*l} + \tanh^2 h^*_1 \frac{1 - z_1^{*l}}{(1 - z^*_1)^2}, \quad b(k) = z_2^{*k} + \tanh^2 h^*_2 \frac{1 - z_2^{*k}}{(1 - z^*_2)^2}, \tag{7.16}$$

for symmetrical case and

$$a(l) = z_1^{*l} + \tanh^2 h^* \frac{1 - z_1^{*l}}{(1 - z^*_1)^2}, \qquad b(k) = z_2^k, \tag{7.17}$$

for asymmetrical one.

Equations $(7.15-17)$ and the way they were derived point on the possibility of obtaining series of asymptotics for free energy per one Ising spin for $2D$ Ising model in the external magnetic field (H). In paper [15] has been shown that vacuum matrix element $S =< 0|V_2^+ V_1^+|0 >$, appearing in (6.16) for $Z_2^+(h)$, for case $(a(l) = y, \quad b(k) = z_2^k)$ is equal to:

$$S = \Gamma^{(h)}(z_2, y) = \prod_{0<q,p<\pi} \Big[1 + z_2^2 + 2z_2 y - 2z_2(1 - y)\cos(p)\Big]^2, \tag{7.18}$$

The above formula may be used to obtain low-temperature asymptotic solution for the free energy $F_{2D}(h)$ per one Ising spin in the thermodynamic limits. Note that the condition $[\tanh^2 h^*/(1 - z^*_1)^2] \to 1$, together with (7.15) is equivalent to $(\exp(-2K_1)(1 - \tanh^2 h) \to 0)$. For given $J_1 = const, H = const$ the above formulated condition is fulfilled for temperature area T, when $h \sim \varepsilon^{-1}, \quad \varepsilon \ll 1$. For that reason, if for instance $([1 - \tanh^2 h^*/(1 - z^*_1)^2] \sim \varepsilon$, then $a(l) = \tanh^2 h^*/(1 - z^*_1)^2 + \sim \varepsilon z_1^{*l}$. Consequently in this case the result (7.18) may be applied. To prove it let us consider Eq. (6.15) for $B_1(q, h)$, expressed by renormalised parameters $(h^*, \ K^*_1)$:

$$B_1(q, h) = \frac{\tanh^2 h^* \frac{\sin q}{1 - \cos q} + 2z^*_1 \sin q}{1 - 2z^*_1 \cos q + z_1^{*2}}, \tag{7.19}$$

where $z^*_1 = \tanh K^*_1$, a h^* i K^*_1 connected with h i K_1 as was shown in (7.15). Moreover, due to identity

$$\frac{z_1^*}{1+z_1^{*2}} = \frac{z_1(1-\tanh^2 h)}{1+2z_1\tanh^2 h + z_1^2},$$

introducing a small parameter $(1-\tanh h) \sim \varepsilon, \quad \varepsilon \ll 1$, and developing $B_1(q,h)$ into series along ε ($z_1^* \sim \varepsilon$), we obtain

$$B_1(q,h) = \frac{(\tanh^2 h^* + 2z_1^*)\sin q}{1-\cos q} + \sim \varepsilon^2$$

Substituting the last expression to (6.22) we come to the formula

$$a(l) = \tanh^2 h^* + 2z_1^*, \tag{7.20}$$

describing $a(l)$ with exactness to the second power of ε ($\sim \varepsilon^2$), i.e. in this approximation $a(l)$ does not depend on l. Finally, substituting in (7.18) y for $a(l)$ expressed by (7.20) we receive in the limiting case the following expression for the free energy $F_{2D}(h)$:

$$-\beta F_{2D}(h) \asymp \ln(2\cosh K_1^*\cosh K_2\cosh h) + \frac{1}{2\pi}\int_0^\pi \ln[1+z_2^2+2z_2(\tanh^2 h^* + 2z_1^*) - 2z_2(1 - \tanh^2 h^* - 2z_1^*)\cos p]dp, \tag{7.21}$$

where h^* and K_1^* depend on h and K_1 according to (7.15). Note that the derived approximation (7.21) may be also applied to the case of comparably strong magnetic field (H) for which $(1-\tanh h) \sim \varepsilon, \quad \varepsilon \ll 1, \quad (T = const)$.

C. High-temperature approximation

In the range of high temperature we impose $(J_{1,2}/k_BT \sim \varepsilon), \quad \varepsilon \ll 1, \quad (J_{1,2} = const, \quad H = const)$, i.e. $z_{1,2} = \tanh K_{1,2} \sim \varepsilon$. In this approximation the bra-vector $< 0|T_2$, expressed in terms of α-operators by (5.45), can be written as:

$$< 0|T_2 \simeq < 0|\exp(z_2 \sum_{n=1}^{N}\sum_{m=1}^{M} \beta_{n,m+1}\beta_{nm}),$$

i.e. expresses in terms of β-operators, multiplying all phase coefficients φ_{nm} (5.15) by bra-vector $< 0|$. It allows for diagonalisation of the operator $T = T_2T_1T_h^*$ in (6.1) and calculation of the vacuum matrix element $< 0|T|0 >$. We will not consider the expressions for the free energy, as the mentioned above approximation seems to be crude approximation and are not of the special interest.

VIII. CONCLUSIONS

The case of infinitely small external magnetic field is very interesting $(h \sim \varepsilon, \quad \varepsilon \ll 1, \quad T = const)$. Because in Eqs. (6.15) and (7.15) the magnetic field h appears in $\tanh^2 h$ function, the computations should be carried out up to the second term (ε^2) inclusive. The presented approach allows for respective calculations, nevertheless they are long and

complicated enough to present them in another paper. We should only like to note here a case connected with calculations of the free energy for the external magnetic field H asymptotic tending to zero, i.e. fulfilling the condition $(h \to 0, \quad N, M \to \infty)$. Neglecting in Eq. (7.16) for $a(l)$ and $b(k)$ terms proportional to $\tanh^2 h^*_{1,2} \sim \tanh^2 h/2$ for $a(l)$ and $b(k)$ we obtain the following asymptotic expressions:

$$a(l) \asymp z_1^{*l}, \quad b(k) \asymp z_2^{*k}, \quad (h \to 0, \quad T = const).$$

In this case we can automatically derive the expression for the free energy, substituting in (7.14) $z_{1,2}$ for $z^*_{1,2}$:

$$-\beta F_{2D}(h \to 0) \asymp \ln 2 + 2\ln(\cosh h/2) +$$
$$\frac{1}{2\pi^2}\int_0^\pi \int_0^\pi \ln[\cosh 2K_1^* \cosh 2K_2^* - \sinh 2K_1^* \cos q - \sinh 2K_2^* \cos p] dq dp,$$

where $\cosh 2K^*_{1,2}$ and $\sinh 2K^*_{1,2}$ are defined by Eqs. (7.15). This is the leading asymptotic term and the latter for $(h = 0)$ given Onsager solution. The procedure is equivalent to considering the asymptotically vanishing magnetic field h in the zero-order approximation, which in the author's opinion is worth analyzing.

The above presented approach to the Lenz-Ising-Onsager problem, on the example of $1D$ and $2D$ Ising model in the external magnetic field may be extended on the $3D$ Ising model in the external magnetic field for the purpose of obtaining the low-temperature approximation. All calculations are then, in fact, same as the ones leading to Eq. (7.21), apart from details connected with dimension of the considered system. The obtained results will be a subject of a future paper.

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REFERENCES

[1] W. Lenz, Phys. ZS., **21**, 613 (1920).
[2] E. Ising, Zs. Physik, **31**, 253 (1925).
[3] L. Onsager, Phys. Rev., **65**, 117 (1944).
[4] L. Onsager, Nuovo Cimento(Suppl.), **6**, 261 (1949).
[5] C.N. Yang, Phys. Rev., **85**, 809 (1952).
[6] T.D. Schultz, D.C. Mattis, and E.H. Lieb, Rev. Mod. Phys., **36**, 856 (1964).
[7] R.J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Ac.Press, Inc.,(1982).
[8] Yu.A. Izyumov, and Yu.N. Skryabin: *Statistical Mechanics of Magnetically Ordered Systems*, Nauka, Moscow, (in russian), (1987).
[9] B. McCoy, and T.T. Wu, *Two Dimensional Ising Models*, Harvard U. Press. Cambridge, Mass., (1973).
[10] M. Gaudin, *La Fonction D'Onde De Bethe*, MASSON, Paris (1983).
[11] Th.M. Liggett: *Interacting Particle Systems*. Springer-Verlag, New York-Tokyo, (1985).
[12] M.S. Kochmański, J.Tech.Phys., **36**, 485 (1995).
[13] J.M. Ziman:*Models of Disorder*, Univ. Press, Cambridge, (1979).
[14] L. Onsager, B. Kaufman: Phys.Rev, **76**, 1232 (1949).
[15] M.S. Kochmański, J.Tech.Phys., **37**, 67 (1996).
[16] K. Huang: *Statistical Mechanics*, J.Wiley and Sons, Inc., New York -London, (1963).
[17] C.J.Thompson,*Classical Equilibrium Statistical Mechanics*, CLARENDON PRESS-OXFORD, (1988).
[18] *Graph Theory and Theoretical Physics*, ed. F. Harary, Ac.Press., (New York),(1967).
[19] P. Jordan, E. Wigner: Z.Physik, **47**, 631 (1928).
[20] L.D. Landau, E.M. Lifschitz: Quantum Mechanics, **III**, 4-th issue, Nauka, Moscow, (in russian), (1989).
[21] G.C. Wick, Phys. Rev., **80**, 268 (1950).
[22] J. Glimm, and A. Jaffe, *Quantum Physics*, Springer-Verlag, New York (1981).
[23] F.Harary, and E.M.Palmer, *Graphical Enumeration*, Ac.Press,(New York),(1973).
[24] W.T. Tutte, *Graph Theory*, Cambridge University Press, (Cambridge), (1984).
[25] C.A. Hurst, and H.S. Green, J.Chem.Phys., **33**, 1059 (1961).
[26] A.M. Dyhne, and Yu.B. Rumer, Us.Phys.Nauk, **75**, 101 (1961), (in russian).
[27] N.V. Vdovitchenko, Zh.Eksp.Teor.Fiz. **47**, 715, (1964) [Sov.Phys. JETP **20**, 477 (1965)].
[28] M.S. Kochmański, J.Tech.Phys., **38**, 73 (1997).
[29] M.S. Kohmański, Zh.Eksp.Teor.Fiz.**111**, 1717 (1997) [JETP **84**, 940 (1997)].
[30] M. Kac, and J.C. Ward, Phys. Rev. **88**, 1332 (1952).
[31] R.P. Feynman, Statistical Mechanics, W. A. Benjamin, Inc., Ad. Book Progr. Reading, Massachusetts (1972).
[32] S. Sherman, Journ. Math. Phys., **1**, 202 (1960); **4**, 1213 (1963).
[33] L.D. Landau, E.M. Lifschitz: Statistical Physics, **Y**, Part 1, 3-th issue, Nauka, Moscow, (in Russian), (1976).

FIGURES

FIG. 1. Examples of some graphs: (a) – a self-avoiding walk; (b)– (f) – Hamilton cycles on a rectangular $(N \times M)$ lattice with equal numbers of vertices and edges (with varying length of steps).

FIG. 2. "Geometry" of transposition relations for α- and β- operators: * – α-operator; $\times$ – β-operator.

FIG. 3. The simplest example of a graph (Euler cycles) giving a contribution to the sum over states $Z(K_1, K_2)$.